Mathematics. — *On Differentiable Linesystems of one Dual Variable.* II. By N. H. KUIPER (Princeton N.Y.) (Communicated by Prof. W. VAN DER WOUDE.)

(Communicated at the meeting of October 30, 1948.)

9. *The strictionsurface of a non-degenerate D-system.*

The strictionpoint at a rule $\mathfrak{A}$ of a regulus is the intersection with the common perpendicular of $\mathfrak{A}$ and a rule infinitesimally near to $\mathfrak{A}$. It is the origin of the Blaschke-system of the regulus, which is also the invariant orthogonal system of the related D-system at the considered line. The strictionsurface of a D-system is the locus of strictionpoints of the reguli in the D-system. It is the locus of origins of first invariant orthogonal systems of the D-system.

We first determine the strictionpoints on the lines $\mathfrak{A}(T) = \mathfrak{A}(t) + \varepsilon \bar{t} \dot{\mathfrak{A}}(t) = \mathfrak{A}(t) + \varepsilon \bar{t} P(t) \mathfrak{A}_1(t)$ (t fixed, $\bar{t}$ variable) of a D-system. These strictionpoints lie in the plane $\mathfrak{A}(t)$, $\mathfrak{A}_2(t)$, and on the lines (resp.)

$$\mathfrak{A}_1(t+\varepsilon \bar{t}) = \mathfrak{A}_1(t) + \varepsilon \bar{t} \dot{\mathfrak{A}}_1(t) = \mathfrak{A}_1 + \varepsilon \bar{t}(-P\mathfrak{A} + Q\mathfrak{A}_2) \quad ((9))$$

These strictionpoints are therefore the points of the line

$$\frac{q\mathfrak{A} + p\mathfrak{A}_2}{V(p^2+q^2)} = \cos\varphi \cdot \mathfrak{A} + \sin\varphi \cdot \mathfrak{A}_2 \quad (T = t).$$

The *strictionsurface of the D-system* is the ruled surface

$$\mathfrak{R}(t) = \frac{q\mathfrak{A} + p\mathfrak{A}_2}{V(p^2+q^2)} = \cos\varphi \cdot \mathfrak{A} + \sin\varphi \cdot \mathfrak{A}_2 \, . \quad . \quad . \quad . \quad (16)$$

The line $\mathfrak{A}_1(T)$ is at the line $\mathfrak{A}(T)$ perpendicular to any of the striction-curves of the reguli in the D-system. Hence $\mathfrak{A}_1(T)$ is a normal of the strictionsurface. $\mathfrak{A}_1(t+\varepsilon\bar{t})$ (t fixed, $\bar{t}$ variable) has a constant direction $\mathfrak{a}_1(t)$. The surfacenormals along a rule of $\mathfrak{R}(t)$ have constant direction and so $\mathfrak{R}(t)$ is developable.

Theorem 7. *The strictionsurface of a non-degenerate D-system $\mathfrak{A}(T)$ is a developable regulus. The D-system $\mathfrak{A}_1(T)$ consists of the normals of this strictionsurface.*

10. *The intrinsic equations of a D-system* [5]).

$\mathfrak{A}(T)$, representing a D-system, was assumed to be analytic:

$$\mathfrak{A}(T) = \sum_{n=0}^{\infty} \frac{d^n \mathfrak{A}(0)}{dT^n} \frac{T^n}{n!} \qquad K_1 < T < K_2.$$

[5]) Compare: BLASCHKE [3], DWINGER [4].

From the equations (9), and equations obtained by derivation with respect to T from (9), $\mathfrak{A}(T)$ is seen to be determined by $\mathfrak{A}(0)$, $\mathfrak{A}_1(0)$, $\mathfrak{A}_2(0)$, $P(T)$ and $Q(T)$ (or $\Phi(T)$) · $(\mathfrak{A}\,\mathfrak{A}_1\,\mathfrak{A}_2)(0)$ can be transformed into any other equally oriented orthogonal system by a motion. The D-system is therefore, but for a motion, determined by the "intrinsic equations"

$$P = P(T),\ Q = Q(T) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

or also by

$$P = P(T),\ \Phi = \Phi(T) \qquad ((9) \text{ and } (11)) \quad . \quad . \quad . \quad . \quad (17)$$

A non-degenerate D-system can be represented with the help of the invariant dual parameter. We get *one* intrinsic equation:

$$\Phi = \Phi(S) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In § 3 we stated that *D-systems are in better analogy with sphere-curves than reguli.* Indeed: a non-degenerate D-system can be characterised intrinsically by one equation (18), analogous to an intrinsic equation of a sphere curve. This is not true for a regulus (with invariant δ; § 6).

11. *The second invariant orthogonal system at a line of a D-system.*

In the theory of threedimensional curves an invariant orthogonal system γ at a point of a curve consists of the tangent, normal and binormal. The first invariant system of § 4 is not strictly analogous to this system. It was possible to choose it because we consider dual *unit* vectors. We call the system Γ analogous to γ, the second invariant orthogonal system at a line of a non-degenerate (assumed) D-system. The three mutually perpendicular and intersecting axes of this system are $(\mathfrak{A} = \mathfrak{A}(S))$

$$\dot{\mathfrak{A}} = \mathfrak{A}_1,\ \mathfrak{A}_2^* = \dot{\mathfrak{A}}_1 / \sqrt{\dot{\mathfrak{A}}_1^2},\ \mathfrak{A}_3^* = \mathfrak{A}_1 \times \mathfrak{A}_2^* \quad . \quad . \quad . \quad . \quad (19)$$

It is related with the first system as follows (11))

$$\left.\begin{aligned} \mathfrak{A}_2^* &= -\sin\Phi \cdot \mathfrak{A} + \cos\Phi \cdot \mathfrak{A}_2 \\ \mathfrak{A}_3^* &= -\cos\Phi \cdot \mathfrak{A} - \sin\Phi \cdot \mathfrak{A}_2\,(= -\mathfrak{S}) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (20)$$

The formulas analogous to the *formulas of* FRENET–SERRET for threedimensional curves are ((11), (19), (20), (9))

$$\left.\begin{aligned} \dot{\mathfrak{A}}_1 &= \phantom{-C\mathfrak{A}_1} \quad C\mathfrak{A}_2^* & &= \phantom{-\operatorname{cosec}\Phi\cdot\mathfrak{A}_1} \quad \operatorname{cosec}\Phi\,\mathfrak{A}_2^* \\ \dot{\mathfrak{A}}_2^* &= -C\mathfrak{A}_1 \qquad\quad + T\mathfrak{A}_3^* & &= -\operatorname{cosec}\Phi\cdot\mathfrak{A}_1 \qquad\quad + \dot{\Phi}\mathfrak{A}_3^* \\ \dot{\mathfrak{A}}_3^* &= \phantom{-C\mathfrak{A}_1} \quad -T\mathfrak{A}_2^* & &= \phantom{-\operatorname{cosec}\Phi\cdot\mathfrak{A}_1} \quad -\dot{\Phi}\mathfrak{A}_2^* \end{aligned}\right\} \quad (21)$$

$C = \operatorname{cosec}\Phi$ is *the dual curvature* of the non-degenerate D-system; $T = \dot{\Phi}$ is *the dual torsion*. We have the formula:

$$\text{dual torsion} = \frac{d}{dS}\operatorname{cosec}^{-1}(\text{dual curvature}) \quad . \quad . \quad . \quad . \quad (22)$$

D-systems for which the dual torsion vanishes, have a constant dual curvature. Their momentanous axis $\mathfrak{S} = -\mathfrak{A}_3^*$ is constant. They are analogous to circles, and each consists of lines which have a constant dual angle with a constant line. If this dual angle is $\Phi = \pi/2 + \varepsilon \cdot 0$ then consists $\mathfrak{A}(S)$ of the perpendiculars to a line. Such a D-system is analogous to a geodesic on the sphere. If we take two non-parallel non-intersecting lines of the "*geodesic D-system*", then there exists a doubly infinite set of lines in the D-system, for any of which in combination with the given lines the triangle-equality holds (6).

(21) can be considered as the system of equations (9) with respect to the D-system $\mathfrak{A}_3^* = -\mathfrak{S}$. The invariant parameter of $\mathfrak{A}_3^*$ is $-\Phi(+K)$. Under the assumption that $\mathfrak{A}_3^*$ is not degenerated, the equations (11) for $\mathfrak{A}_3^*$ become ((see (21))

$$\left.\begin{aligned} \left(\frac{d\mathfrak{S}}{d\Phi}=\right)\frac{d\mathfrak{A}_3^*}{d-\Phi} &= \mathfrak{A}_2^* \\ \frac{d\mathfrak{A}_2^*}{d-\Phi} &= -\mathfrak{A}_3^* + \frac{\operatorname{cosec}\Phi}{\dot{\Phi}}\cdot\mathfrak{A}_1 \\ \frac{d\mathfrak{A}_1}{d-\Phi} &= -\frac{\operatorname{cosec}\Phi}{\dot{\Phi}}\cdot\mathfrak{A}_2^* \end{aligned}\right\} . \quad (23)$$

The formulas (11) for the enveloped D-system $\mathfrak{S} = -\mathfrak{A}_3^*$ are

$$\left.\begin{aligned} \frac{d\mathfrak{S}}{d\Phi} &= \mathfrak{S}_1 \\ \frac{d\mathfrak{S}_1}{d\Phi} &= -\mathfrak{S} - \frac{\operatorname{cosec}\Phi}{\dot{\Phi}}\cdot\mathfrak{S}_2 \\ \frac{d\mathfrak{S}_2}{d\Phi} &= \frac{\operatorname{cosec}\Phi}{\dot{\Phi}}\,\mathfrak{S}_1 \end{aligned}\right\} \quad . \; . \; . \quad (23')$$

The angle $\Phi^{(2)}$ between a rule of the "enveloped D-system" $\mathfrak{S}$ and its momentanous axis $\mathfrak{S}^{(2)}$, arclength of $\mathfrak{S}^{(2)}$, is obtained from (23′) and (11):

$$\left.\begin{aligned} \operatorname{cotg}\Phi^{(2)} &= -\operatorname{cosec}\Phi\Big/\frac{d\Phi}{dS} \\ \operatorname{tg}\Phi^{(2)} &= \frac{d\cos\Phi}{dS} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \quad (24)$$

(24) is analogous to the following formula for a Euclidean plane curve

$$r' = r\,\frac{dr}{ds}$$

s is the arclength of the curve, r is the radius of curvature of the curve, r' is the radius of curvature of the evolute of the curve.

From (20), (23), (23′) we get:

$$\left.\begin{aligned} \mathfrak{S} = \cos\Phi\cdot\mathfrak{A} + \sin\Phi\cdot\mathfrak{A}_1, \quad \mathfrak{S}_1 = -\sin\Phi\cdot\mathfrak{A} + \cos\Phi\cdot\mathfrak{A}_2 \\ \mathfrak{A} = \cos\Phi\cdot\mathfrak{S} - \sin\Phi\cdot\mathfrak{S}_1 \end{aligned}\right\} \quad (25)$$

12. *Osculating D-systems.*

In general one can define the n-th enveloped $\mathfrak{S}^{(n)}(T)$ of a D-system $\mathfrak{A}(T)$, if it exists, inductively by: $\mathfrak{S}^{(k+1)}(T)$ is the enveloped D-system with respect to the developed D-system $\mathfrak{S}^{(k)}(T) \cdot k = 0, 1, \dots n-1$. A generalisation of the formula (24) is

$$\operatorname{tg} \Phi^{(k+1)} = \frac{d \cos \Phi^{(k)}}{d\, \Phi^{(k-1)}} = -\sin \Phi^{(k)} \, \frac{d\, \Phi^{(k)}}{d T} \Big/ \frac{d\, \Phi^{(k-1)}}{d T} \quad . \quad . \quad . \quad (26)$$

$\Phi^{(i)}(T)$ is the dual arclength of $\mathfrak{S}^{(i)}(T)$.

The existence of the n-th enveloped at a line of a D-system, admits us to construct a rather simple D-system which osculates of order n with the given D-system at the considered line.

Two non-degenerate D-systems $\mathfrak{A}(S)$ and $\mathfrak{B}(S)$ are said to osculate of order n at the line $S = 0$, when (compare SABAN [9])

$$\mathfrak{A} = \mathfrak{B} \quad \text{and} \quad \frac{d^i\, \mathfrak{A}}{d\, S^i} = \frac{d^i\, \mathfrak{B}}{d\, S^i} \; i = 1, \dots, n+1 \quad S = 0,$$

equivalent to:

$$\mathfrak{A} = \mathfrak{B},\; \mathfrak{A}_1 = \mathfrak{B}_1,\; \Phi_a = \Phi_b,\; \frac{d^i\, \Phi_a}{d S} = \frac{d^i\, \Phi_b}{d S}\; ; i = 1, \dots, n-1 \quad S = 0,$$

equivalent to ((25) and equations obtained by differentiation of (25)):

$$\mathfrak{A} = \mathfrak{B},\; \mathfrak{A}_1 = \mathfrak{B}_1,\; \Phi_a = \Phi_b,\; \Phi_a^{(i)} = \Phi_b^{(i)}\; ; i = 2, \dots, n \quad S = 0,$$

equivalent to:

$$\mathfrak{A} = \mathfrak{B},\; \mathfrak{S}_a^{(i)} = \mathfrak{S}_b^{(i)}\; ; \; i = 1, \dots, n \;\; S = 0.$$

A rather simple D-system which osculates of order n with a given D-system $\mathfrak{A}(S)$ with existing n-th developable at the line $S = 0$, is the D-system $\mathfrak{B}(S)$ for which:

$$\mathfrak{B}(0) = \mathfrak{A}(0),\; \mathfrak{S}_b^{(i)}(0) = \mathfrak{S}_a^{(i)}(0)\; ; i = 1, \dots, n-1,\; \mathfrak{S}_b^{(\dot{n})}(S) = \mathfrak{S}_a^{(n)}(0),$$

hence also

$$\Phi_b^{(i)}(0) = \Phi_a^{(i)}(0)\; ; i = 1, \dots, n-1,\; \Phi_b^{(\dot{n})}(S) = \Phi_a^{(n)}(0).$$

Examples.

1) *A simple D-system which osculates of order* 1 with a non-degenerate D-system $\mathfrak{A}(S)$ at the line $S = 0$, consists of the lines that make the same dual angle $K = \Phi(0)$ with $\mathfrak{S}(0)$ as $\mathfrak{A}(0)$.

Putting:

$$\mathfrak{S}(0) = (0, 0, 1), \quad \mathfrak{A}(0) = (0, \sin K, \cos K)$$

the required D-system is found to be:

$$\mathfrak{B}(S) = \left(\sin \frac{S}{\sin K} \cdot \sin K,\;\; \cos \frac{S}{\sin K} \cdot \sin K,\;\; \cos K \right) \quad . \quad (27)$$

2) $\mathfrak{A}(T)$ be a D-system with a non-degenerate enveloped D-system. We want to construct *a simple D-system which osculates of order* 2 with $\mathfrak{A}(T)$ at the line $T = 0$.

Let $\mathfrak{S}^{(2)}(0) = (0, 0, 1)$, $\mathfrak{S}(0) = (0, \sin \Lambda, \cos \Lambda)$, $\Lambda = \Phi^{(2)}(0)$.
The dual angle between $\mathfrak{A}(0)$ and $\mathfrak{S}(0)$ be K.
Then is the enveloped D-system of the required D-system:

$$\mathfrak{C}(\Phi) = \left(\sin \frac{\Phi - K}{\sin \Lambda} \cdot \sin \Lambda, \quad \cos \frac{\Phi - K}{\sin \Lambda} \sin \Lambda, \quad \cos \Lambda\right) \quad . \quad (28)$$

Φ is invariant parameter of $\mathfrak{C}$ and is also the dual angle between $\mathfrak{B}(\Phi)$ and $\mathfrak{C}(\Phi)$, where $\mathfrak{B}(\Phi)$ is the required D-system.
By differentiation of (28) we get:

$$\mathfrak{C}_1(\Phi) = \left(\cos \frac{\Phi - K}{\sin \Lambda}, \; -\sin \frac{\Phi - K}{\sin \Lambda}, \; 0\right).$$

The required D-system is found from (25):

$$\left.\begin{aligned} &\mathfrak{B}(\Phi) = \cos \Phi \cdot \mathfrak{C}(\Phi) - \sin \Phi \cdot \mathfrak{C}_1(\Phi) = \\ &\left(\cos \Phi \cdot \sin \Lambda \cdot \sin \frac{\Phi - K}{\sin \Lambda} - \sin \Phi \cdot \cos \frac{\Phi - K}{\sin \Lambda}, \right. \\ &\qquad \left. \cos \Phi \cdot \sin \Lambda \cdot \cos \frac{\Phi - K}{\sin \Lambda} \cdot \sin \Phi \cdot \sin \frac{\Phi - K}{\sin \Lambda}, \; \cos \Phi \cos \Lambda\right) \end{aligned}\right\} . \quad (29)$$

The formula is also applicable when $\mathfrak{A}(T)$ is degenerate at $T = 0$ (the enveloped D-system of $\mathfrak{A}(T)$ however non-degenerate: $K = 0 + \varepsilon \bar{k}$, $\Lambda \neq 0 + \varepsilon \bar{\lambda}$).

13. *A formula of* EULER–SAVARY *and the analogue in linegeometry.*

A formula (f), like the formula of EULER–SAVARY (31) in the geometry on the sphere, reduces to a set (i) of real identities in real variables, when the involved entities (e) are replaced by their definitions (d). The functional identities (i), hold equally well for differentiable functions of dual variables (§ 1; KUIPER [7] Ch. 1), if only (assumption Z) we exclude those values of the dual variables for which a division by a zero-divisor would occur in the identities. Under assumption Z we can define entities (E) in linegeometry by definitions (D) analogous to (d). The defining formulas (D) are the differentiable dual continuations of the formulas (d). The formula (F), analogous by differentiable continuation to (f), holds true for the entities (E).

So here we have a method to construct, and at the same time to prove, formulas (theorems) on D-systems.

Examples of entities (E) are the dual curvature $\sin \Phi$ (21) and the momentanous axis $\mathfrak{S}$ of a D-system. An example of a formula is (24). According to the theory above, (24) follows without further proof from the formula for a curve on a unitsphere

$$\operatorname{tg} \varphi^{(2)} = \frac{d \cos \varphi}{d s} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (30)$$

s is the arclength in radials; tg φ, resp. tg $\varphi^{(2)}$, is the geodesic curvature of the curve, resp. of the evolute of the curve. (The evolute or developable is defined by a formula analogous to (13)).

We will conclude with another application of the theory, namely to the formula of EULER–SAVARY on the sphere:

$$\operatorname{cotg} \varphi_1 - \operatorname{cotg} \varphi = -(\operatorname{cotg} \varphi_f - \operatorname{cotg} \varphi_m) \operatorname{cosec} \psi. \quad . \quad . \quad (31)$$

This formula is related with the motion of a mobile unitsphere s_m containing a curve c_1, over a fixed concentric unitsphere s_f. The moving curve c_1 envelopes a curve c of s_f. Those momentanous invariant points of s_m, with respect to which the velocity of s_m is positive (compare the remark after (15); at each moment we have the choice between two invariant points), generate a curve c_f on s_f and a curve c_m on s_m. At the moment under consideration be p the invariant point of the motion, q the tangent point of c_1 and c, $\psi + \pi/2$ is the positive angle between the tangent at p to c_f and the tangent at q to c, both tangents equipped with the direction of increasing arclength, as seen from the direction of the outside-sphere-normal at p. φ_1, φ, φ_f, φ_m are the arclengths from p to the curvature-centres of c_1, c, c_f, c_m.

From (31) follows the analogous formula concerning linegeometry:

$$\operatorname{cotg} \Phi_1 - \operatorname{cotg} \Phi = -(\operatorname{cotg} \Phi_f - \operatorname{cotg} \Phi_m) \operatorname{cosec} \Psi, \quad . \quad . \quad (32)$$

equivalent to the two real formulas [6])

$$\left\{ \begin{array}{l} \operatorname{cotg} \varphi_1 - \operatorname{cotg} \varphi = -(\operatorname{cotg} \varphi_f - \operatorname{cotg} \varphi_m) \operatorname{cosec} \Psi \\ \left(\dfrac{\bar{\varphi}_1}{\sin^2 \varphi_1} - \dfrac{\bar{\varphi}}{\sin^2 \varphi} \right) \sin \psi = - \dfrac{\bar{\varphi}_f}{\sin^2 \varphi_f} + \dfrac{\bar{\varphi}_m}{\sin^2 \varphi_m} + \bar{\psi} \operatorname{cotg} \psi \, (\operatorname{cotg} \varphi_f - \operatorname{cotg} \varphi_m) \end{array} \right.$$

$$\left(\operatorname{cotg} \Phi = \operatorname{cotg} \varphi + \varepsilon \bar{\varphi} \, d \operatorname{cotg} \varphi / d\varphi = \operatorname{cotg} \varphi - \varepsilon \frac{\bar{\varphi}}{\sin^2 \varphi} \quad (2)\right)$$

This formula is related with the dual motion of a mobile Euclidean three-dimensional space S_m, containing a D-system ${}^1\mathfrak{A}$, with respect to a fixed space S_f. The moving ${}^1\mathfrak{A}$ is at each moment T at one of its lines "tangent" to a D-system $\mathfrak{A}$ in S_f (${}^1\mathfrak{A} = \mathfrak{A}$, ${}^1\mathfrak{A}_1 = \mathfrak{A}_1$). The moving ${}^1\mathfrak{A}$ "envelopes" $\mathfrak{A}$. The momentanous axes of the dual motion generate a D-system ${}^f\mathfrak{A}$ in S_f and a D-system ${}^m\mathfrak{A}$ in S_m. The dual motion can be considered as a development of the D-system ${}^m\mathfrak{A}$ along the D-system ${}^f\mathfrak{A}$. ${}^f\mathfrak{A}(0)$ is the momentanous axis at a moment under consideration ($T = 0$). $\mathfrak{A}(0)$ is the line at which ${}^1\mathfrak{A}$ and $\mathfrak{A}$ are tangent. $\pi/2 + \Psi$ is the dual angle between ${}^f\mathfrak{A}_1(0)$ and $\mathfrak{A}_1(0)$.

Φ_1, Φ, Φ_f, Φ_m are the dual angles between ${}^f\mathfrak{A}(0)$ and the momentanous axes of ${}^1\mathfrak{A}$, $\mathfrak{A}$, ${}^f\mathfrak{A}$, ${}^m\mathfrak{A}$.

If we restrict the dual variable, time T, to moments $T = T(u) = t(u) + \varepsilon \bar{t}(u)$ $(d\, t(u) / du \neq 0)$, then u can be considered as an

[6]) Compare: VAN HAASTEREN [8] p. 59 formula 66; DISTELLI [2] p. 305 form. 74.

ordinary time-variable with respect to which an ordinary motion is determined. The results we may get are then stated in terms of reguli.

These results are also obtained in [2] [5] [8].

BIBLIOGRAPHY.

1. E. STUDY, Geometrie der Dynamen (1903).
2. J. WOLFF, Dynamen beschouwd als duale Vectoren. Thesis, Amsterdam (1908).
3. M. DISTELI, Ueber das Analogon der Savaryschen Formel und Konstruktion in der kinematischen Geometrie des Raumes. Zeitschr. f. Math. u. Phys., **62**, 261—309 (1914).
4. W. BLASCHKE, Vorlesungen ueber Differential-Geometrie 1 Kap 9 (1930).
5. PH. DWINGER, Differentiaal-meetkundige beschouwingen over lijnenstelsels. Thesis Leiden (1938).
6. R. GARNIER Extension de la formule de Savary au mouvement le plus général d'un solide. Ann. sc. Ecole Normale Sup. 3—57, p. 113—200 (1940).
7. KERIM ERIM, Die höheren Differentialelemente einer Regelfläche und einer Raumkurve. Rev. Istambul A 10—14 (1945).
8. L. BIRAN, *a.* Les surfaçes réglées étudiées en analogie avec les courbes gauches. Revue de la Fac. d. Sc. de l'Univ. d'Istanbul A, 10—1—4 (1945). *b.* Extension des notions de développée et de développante et leurs images sur la sphère dualistique. Id. A, 11—1—2 (1946). *c.* Mouvement à un paramètre Id. A, 12—3 (1947). *d.* Extension de la Construction de Savary à l'espace réglé, Id. A, 12—2 (1948).
9. N. H. KUIPER, Onderzoekingen over lijnenmeetkunde. Thesis, Leiden (1946).
10. A. VAN HAASTEREN, Over de formule van Euler–Savary en haar uitbreidingen in de cinematische meetkunde van de Euclidische ruimte en het niet-euclidische vlak. Thesis Leiden 1947.
11. G. SABAN, Raccordement d'ordre élevé de deux surfaçes réglées. Rev. Istanbul A, 13—1 (1948).