Mathematics. — *On the symbolical method.* I. By E. M. BRUINS. (Communicated by Prof. L. E. J. BROUWER.)

(Communicated at the meeting of November 27, 1948.)

§ 1. *Introduction.*

Denoting the homogeneous coordinates of a point X in $(n-1)$-dimensional projective space G_n by $(x_1, x_2, \dots, x_n)$ and the space-coordinates of a hyperplane U' by $(u_1', u_2', \dots u_n')$ one can use the short notations

$$(xy \dots z), \quad (u'v' \dots w'), \quad (u'x)$$

for determinants whose columns are the coordinates of the points $X, Y, \dots Z$, resp. of the spaces $U', V_{,}, \dots W'$ and the linear form

$$u_1'x_1 + u_2'u_2 + \dots u_n'x_n.$$

The first fundamental theorem of the theory of invariants asserts, that every rational projective invariant of a system of points and spaces can be expressed in polynomia of these symbols only. The theorem holds also if the points and spaces are merely symbolical, i.e. stand for variables, which are transformed cogredient or contragredient.

Between these symbols some relations are evident:

I. As the determinant

$$\begin{vmatrix} a_1 \dots a_n\,(au') \\ b_1 \dots b_n\,(bu') \\ \vdots \\ d_1 \dots d_n\,(du') \end{vmatrix} = \text{det. } a_1\, b_2 \dots (du'),$$

vanishes, we have

$$(ab \dots c)(du') \equiv (db \dots c)(au') - (da \dots c)(bu') \dots \pm (dab \dots)(cu').$$

This identity shows, that $(n+1)$ points in G_n are linearly dependent and furnishes the homogeneous coordinates of d, the fundamental simplex being $a, b, \dots c$.

II. $(ab \dots c)(u'v' \dots w') \equiv \text{det. } (au')(bv') \dots (cw')$.

III. Denoting the point of intersection of $(n-1)$ spaces $u', v', \dots w'$ by $x_i = (u'v' \dots w')_{\bar{i}}$ we have

$$(ab \dots cx) \equiv (ab \dots c\ (u'v' \dots w')) \equiv \text{det. } (au')(bv') \dots (cw').$$

The second fundamental theorem of the theory of invariants asserts, that every relation between invariants can be deduced with identities of this form only. In order to avoid confusion: in G_2 the determinants are written (ab), the linearform a_x and the identities are then

$$(ab)c_x \equiv (cb)a_x - (ca)b_x,$$
$$(ab)(xy) \equiv a_x\, b_y - a_y\, b_x.$$

This symbolical method has some disadvantages which can clearly be shown comparing the following examples.

1. The proof of DESARGUES-theorem for $n=3$, $n=4$.
$n=3$.

Two triangles being a, b, c and α, β, γ and denoting the opposite sides of $a, \alpha, \ldots$ with $a', \alpha', \ldots$ we have

$$((a'\alpha')(b'\beta')(c'\gamma')) \equiv (((bc)(\beta\gamma))((ca)(\gamma\alpha))((ab)(\alpha\beta))) \equiv$$
$$\equiv ((bc)(ca)(\gamma\alpha))((\beta\gamma)(ab)(\alpha\beta)) - ((\beta\gamma)(ca)(\gamma\alpha))((bc)(ab)(\alpha\beta)) \equiv$$
$$\equiv (abc)(\alpha\beta\gamma)[(c\gamma\alpha)(b\beta a) - (c\gamma a)(b\beta\alpha)] \equiv (abc)(\alpha\beta\gamma)((c\gamma)\, b\beta)(a\alpha)).$$

This shows, that if a, b, c and α, β, γ are not collinear triples, then the collinearity of the points of intersection of corresponding sides is equivalent to the concurrence of the lines joining corresponding vertices.

$n=4$.

If the lines of intersection of corresponding planes of two tetraedra a, b, c, d and $\alpha, \beta, \gamma, \delta$ are coplanar, the lines joining the corresponding vertices are concurrent. We have

$$(d\delta\, a\, \alpha) \equiv ((a'b'c')(\alpha'\beta'\gamma')(b'c'd')(\beta'\gamma'\delta')) \equiv$$
$$\equiv (a'b'c'd')[(b'\beta'\gamma'\delta')(c'\alpha'\beta'\gamma') - (c'\beta'\gamma'\delta')(b'\alpha'\beta'\gamma')] \equiv$$
$$\equiv (a'b'c'd')(\alpha'\beta'\gamma'\delta')(b'\beta'c'\gamma'),$$

which proves the theorem.

In these two deductions we find the highest degree of symmetry and no further simplification seems possible or necessary.

2. We ask for the equation of a conic through five given points a, b, c, d, e in G_3. The pencil of conics through a, b, c, d can be written

$$(abx)(cdx) + \lambda(acx)(bdx) = 0$$

To obtain a final result, λ has to be calculated from

$$(abe)(cde) + \lambda\,(ace)(bde) = 0$$

and so

$$(ace)(bde)(abx)(cdx) - (abe)(cde)(acx)(bdx) = 0$$

is *an* equation for this conic. The lefthand side is, however, invariant under the interchange of a and d or b and c. Therefore we obtain a large number of equivalent left-hand-sides by permutation of a, b, c, d, e, the equivalence of which can be proved by the identities quoted above. We can remark, that the left-hand-side of the equation is the determinant formed from the points of intersection of opposite sides of the hexagon $a\,c\,e\,b\,d\,x$, as this determinant is

$$(((ac)\ (bd))\ ((ce)\ (dx))\ ((eb)\ (xa))),$$

so the equation above proves PASCAL's theorem.

The equation of the conic is in each of the variables of the second degree, just as it has to be, and the only complication that remains is here the great number of equivalent formulae which may cause a difficulty: to choose the right identities necessary to prove the equivalence (in a most direct way).

3. We ask for the equation of a PASCAL-line of the conic a, b, c, d, e, f *e.g.* that joining the points of intersection of ac, bd and ce, df.

We have then

$$(((ac)(bd)) \quad ((ce)(df)) \quad x) \equiv$$
$$(ace)(cdf)(bdx) + (dce)(bdf)(acx) = 0.$$

Here again we obtain several equivalent lefthand-sides of the first degree in a, f, b, e of the second degree in c, d. As the PASCAL-lines describe a pencil of lines if one of the points moves along the conic, the other five remaining fixed there are one c and one d "too much". Here the situation is more serious: by identical transformations the degree of each of the variables cannot be changed. The only possibility to eliminate superfluous elements is to transform the equation so as to split off bracket-factors containing all superfluous elements. But with two elements c, d no not vanishing bracketfactor can be formed: *it is impossible* to eliminate the superfluous c, d in the above equation of the PASCAL-line by identical transformations.

In order to remove the dissymmetry we must have a method which splits up the ternary brackets into symbols containing *at most* two elements.

In higher dimensional spaces the abundance of equivalent forms becomes overwhelming and dissymmetries occur very often.

The splitting up of the n-air bracketfactors can be obtained from a normalcurve in G_n. Be the curve in parametric representation

$$P_t \equiv (AU')\, a_t^{n-1} = 0,$$

then the bracketfactor

$$(P_i\, P_k \dots P_l\, P_m) \equiv (AB \dots CD)\, a_i^{n-1}\, b_k^{n-1} \dots c_l^{n-1}\, d_m^{n-1}.$$

As this form vanishes for $i = k$, $i = l$, ... $l = m$ we have

$$(P_i P_k \dots P_l P_m) \equiv \text{const. } I.\ (ik)(il) \dots (im) \dots (lm)$$

where

$$I = (AB \dots CD)\ (ab) \dots (cd)$$

is the invariant, which vanishes if the curve lies in a G_k, $k \leq n-1$.

A normal curve defines a polarity (incident for $n =$ even, non-incident for $n =$ odd) and specifying the hyperplanes by the parameters of the points of intersection with the normalcurve and the points by the parameters of the points of intersection of the normalcurve with the polar-hyperplanes we can develop a symbolical method in which all the advantages are preserved but new possibilities created by the breaking up of the determinant factors.

§ 2. $(AU')a_t^2$.

If $A, \dots U'$ are ternary symbols, $a, \alpha, \dots$ binary, we have a parametric representation of a conic in

$$P_t \equiv (AU')\, a_t^2 = 0.$$

For a given t we have the equation of the point of the conic; a given line U' contains two points, which coincide if

$$(AU')\ (BU')\ (ab)^2 = 0,$$

the equation of the conic in linecoordinates.

A cord of the conic, joining the points i, k has the equation

$$(ABX)\, a_i^2\, b_k^2 \equiv (ABC)\,(ab)\, a_i\, b_k\, (ik) = 0.$$

Dividing by $(ik) \neq 0$ we find the equation of the tangent in the point t as

$$(ABX)\ (ab)\ a_t b_t = 0,$$

the lefthand-side of which we can replace by

$$(A'X)\, \alpha_t^2.$$

The equation of the conic in pointcoordinates is therefore

$$(\Omega' X)^2 \equiv (A'X)\ (B'X)\ (\alpha\beta)^2 = 0.$$

The conic degenerates if and only if three points are collinear i.e. if

$$A_1\, a_t^2 \qquad A_2\, a_t^2 \qquad A_3\, a_t^2$$

are linearly dependent. This means that the corresponding binary quadratic forms represent three pairs of an involution, which gives

$$I = (ABC)\ (ab)\ (bc)\ (ca) = 0.$$

Theorem:

The complete system of $(AU')a_t^2, \varphi_t^m, \psi_t^n, \dots$ consists of

$$I, (AU')\, a_t^2,\ (A'X)\, \alpha_t^2,\ (A'X)\,(B'X)\,(\alpha\beta)^2,\ (AU')\, BU')\,(ab)^2$$

and the comitants which are generated from the complete system of

$$m_t^2,\ \varphi_t^m,\ \psi_t^n, \dots$$

by replacement of m_t^2 by $(AU')a_t^2$ or $(A'X)\alpha_t^2$

Proof:

a) $(ABC) \dots$ is reducible to $I \dots$.

The form $(ABC)\ (ab)^2 \dots \equiv 0$.

The form $(ABC)\ (ab) a_x b_y c_z c_t$ is a polar form of $(ABC)\ (ab) a_x b_y c_z^2$ Changing A, B, C in B, C, A; C, A, B we have

$$(ABC)\,(ab)\, a_x\, b_y\, c_z^2 \equiv \tfrac{1}{3}\,(ABC)\,[(ab)\, a_x\, b_y\, c_z^2 + (bc)\, b_x\, c_y\, a_z^2 + (ca)\, c_x\, a_y\, b_z^2] \equiv$$
$$\equiv \tfrac{1}{3}\,(ABC)\,(ab)\,(bc)\,(ca) \cdot (zx)\,(yz).$$

The form $(ABC) a_x a_y b_z b_t c_u c_v \dots$ is a polar form of $(ABC) a_i^2 b_k^2 c_l^2 \equiv \equiv \frac{1}{3} I \cdot (ik)\ (kl)\ (li)$.

b) $K \equiv (ABX)\ (ab)\ (ac)\ b_x c_y (CU') \ldots$ is reducible to $I \ldots$.

We have, transforming the ternary symbols

$K \equiv (CBX)(ab)(ac)b_x c_y(AU') - (CAX)(ab)(ac)b_x c_y(BU')$ + reducible forms.

Interchanging A, C in the first and B, C in the second term we have

$$K \equiv (ABX)(CU')[(cb)(ca)\, b_x\, a_y + (ac)(ab)\, c_y\, b_x] \ldots + I \cdot (XU') \ldots$$
$$\equiv (ABX)(CU')(ab)(bc)(ca)(xy) + I \cdot (XU') \ldots \equiv I \cdot [(XU') \ldots + (XU') \ldots].$$

The only possible irreducible comitants containing $(ABX) \ldots$ are therefore of the form $(ABX)\ (ab) a_y b_z \ldots \equiv (A'X) \alpha_y \alpha_z \ldots$ in which y, z are *not* connected with $a_x^2 \ldots$, and $(A'X)\ (B'X)\ (\alpha\beta) \ldots$ which is reducible to $(A'X)\ (B'X)\ (\alpha\beta)^2$; for interchanging A and B we can always obtain the factor $(ABX)\ (ab) \ldots$. *q.e.d.*

According to these formulae we have:

1. $(A'B'U')(\alpha\beta)\, \alpha_\lambda \beta_\lambda \equiv {}^1/_2\, ((AB)B'U')(ab)\, [(a\beta)\, b_\lambda + (b\beta)\, a_\lambda]\, \beta_\lambda \equiv$
$$\equiv - \tfrac{1}{6} I \cdot (AU')\, a_\lambda^2.$$

This identity shows, that the line-equation obtained from the point-equation of a conic is the same as the original equation, provided that the conic is not degenerated. The importance of this identity lies in the fact that we can interchange point- and line-equation by a simple change of A, A' and a, α.

2. The bracketfactor $(P_i P_k P_l)$ of three points on the conic is split up in a cycle

$$\langle i\, k\, l \rangle = (ik)\ (kl)\ (li),$$

apart from a factor $\frac{1}{3} I$.

3. The equation of the line joining two points, the poles of the cords $P_p P_q$ and $P_r P_s$ is

$$(ABX)\, a_p a_q\, b_r b_s \equiv \tfrac{1}{2} (A'X)\, [(qr)\, \alpha_p \alpha_s + (ps)\, \alpha_q \alpha_r] \equiv$$
$$\equiv \tfrac{1}{2} (A'X)\, [(pr)\, \alpha_q \alpha_s + (qs)\, \alpha_p \alpha_r] = 0.$$

4. The linearform of the cord $P_p P_q$ and the point, being the pole of $P_r P_s$ is

$$(A'B) \alpha_p \alpha_q b_r b_s \equiv -\tfrac{1}{6} I \cdot [(ps)\ (qr) + (qs)\ (pr)].$$

This indentity shows, that if the cord $P_p P_q$ contains the pole of $P_r P_s$, then the cross-ratio $(pqrs) = -1$, provided $I \neq 0$.

5. $(A'C)\ (B'X)\ (\alpha\beta)^2 c_p c_q \equiv - \frac{1}{3} I (B'X) \beta_p \beta_q$,

which expresses the fact, that the polar line of the pole of the cord $P_p P_q$ is the cord itself, provided $I \neq 0$, and that the HESSIAN of the conic is apart from a constant I^2.

6. $(A'B)\ (B'C)\ (\alpha\beta)^2 b_p b_q c_r c_s =$ const. $I^2\, [(pr)\ (qs) + (qr)\ (ps)]$, which indicates that the pole of the cord $P_p P_q$ is conjugated to the pole of the cord $P_r P_s$ when the crossratio $(pqrs) = -1$, provided $I \neq 0$.

Denoting the cycle $(ik)\ (kl)\ (lm)\ \dots\ (qi)$ by

$$\langle\ iklm \dots q\ \rangle$$

we have

$$\langle iklm \dots q\rangle \equiv (\mathbf{ik})(kl)(\mathbf{lm})\dots(qi) \equiv -(lk)^2\langle im\dots q\rangle - \langle ilkm\dots q\rangle$$

$$\langle iklm\dots q\rangle \equiv \langle klm\dots qi\rangle.$$

From this follows, that a cycle with an even number of bracketfactors is reducible to a sum of products of squares of these factors, as is clear from the iteration of the first formula, combined with cyclical permutation of the indices in the last cycle.

For cycles with six-brackets we have also

$$\langle iklmnp\rangle - \langle iknplm\rangle \equiv \langle inp\rangle\langle klm\rangle - \langle knp\rangle\langle ilm\rangle$$

as follows writing out the cycles and transforming identically the first and fourth brackets.

Because of the ternary interpretation it is evident that:

$$\langle inp\rangle\langle klm\rangle - \langle knp\rangle\langle ilm\rangle + \langle kip\rangle\langle nlm\rangle - \langle kin\rangle\langle plm\rangle \equiv 0.$$

Standardequations and standardforms

The complete system of the form $(AU')a_t^2$ and a system of points with parameters $i, k, \dots$ contains the simultaneous invariant forms

$$(AU')a_i a_k \qquad (A'X)\alpha_i\alpha_k$$

only. As a standardequation of the pole of the cord P_iP_k we use

$$\underline{\{ik\} = \frac{(AU')\,a_i a_k}{(ik)} = 0}$$

and we denote the equation of the cord itself by

$$\underline{[ik] = \frac{(A'X)\,\alpha_i\alpha_k}{(ik)} = 0.}$$

We then have

$$\underline{\{ik\} \equiv -\{ki\}} \quad ; \quad \underline{[ik] = -[ki]}.$$

The point of intersection of the cords $[ik] = 0$ and $[lm] = 0$ is then:

$$\{im\} + \{kl\} = 0 \quad \text{or} \quad \{il\} + \{km\} = 0.$$

Denoting the standardequation with

$$\underline{P_{ik,lm} \equiv \{im\} + \{kl\}}$$

we have:

$$P_{ik,lm} \equiv P_{ki,ml} \equiv -P_{ml,ki} \equiv -P_{lm,ik}$$

$$(kl)(im)\,P_{ik,lm} \equiv (il)(km)\,P_{ik,ml}.$$

The linear form of the cord $[ik]$ and the point $\{lm\}$ is apart from a factor $-\frac{1}{6}l$:

$$(il)\ (km) + (im)\ (kl).$$

§ 3. Three points on a conic.

Be the three points P_i, P_k, P_l. Squaring the identity

$$(ki)a_l \equiv (li)a_k - (lk)a_i$$

we obtain at first

$$\langle ikl\rangle \{ik\} \equiv -\tfrac{1}{2}((li)^2 a_k^2 + (lk)^2 a_i^2 - (ik)^2 a_l^2).$$

Joining the point P_i with the pole of the cord $[kl] = 0$ we have

$$t_{i,kl} = [ki] + [li].$$

From this is evident

$$t_{i,kl} + t_{k,li} + t_{l,ik} \equiv 0,$$

which shows that the three lines are concurrent, the theorem of BRIANCHON for the triangle. The equation of the BRIANCHON-point Δ_{ikl} is obtained intersecting

$$[ki] + [li] = 0 \quad \text{and} \quad [lk] + [ik] = 0$$

which gives immediately

$$(lk)^2 P_i + (ik)^2 P_l + (il)^2 P_k = 0.$$

In virtue of the relation quoted above we obtain permutating cyclically and dividing by $< ikl >$ as a standardform of Δ_{ikl}:

$$\Delta_{ikl} \equiv \{ik\} + \{kl\} + \{li\}.$$

According to the formula 6. we have immediately

$$(\Omega' \Delta_{ikl})^2 \equiv \text{const. } I^2,$$

which is independent of i, k, l.

§ 4. Four points on a conic.

The evident relations for $P_{ik,lm}$ given above can be completed by

$$\begin{aligned} \langle iklm\rangle P_{ik,lm} &\equiv -\langle klm\rangle P_i + \langle ilm\rangle P_k \\ &\equiv \quad \langle ikm\rangle P_l - \langle ikl\rangle P_m. \end{aligned}$$

Proof:

$$\begin{aligned} \langle iklm\rangle P_{ik,lm} &\equiv \langle iklm\rangle \{im\} + \langle iklm\rangle \{kl\} \equiv \\ &\equiv (lm)[-(ik)(kl)\, a_i\, a_m + (ik)(mi)\, a_k\, a_l] \equiv \\ &\equiv -\langle klm\rangle a_i^2 + \langle ilm\rangle a_k^2, \end{aligned}$$

omitting the ternary factor for sake of simplified notation. The second half of the theorem follows transforming the other factors.

As an immediate consequence of the standardformula we have in

$$P_{ik,lm} \equiv \{im\} + \{kl\} = 0 \qquad P_{il,km} \equiv \{im\} + \{lk\} = 0$$

the equations of the diagonalpoints of the tetragon. It is evident from these, that the diagonalpoints are collinear and harmonical with the poles of the cords $[im] = 0$, $[kl] = 0$.

Moreover we have

$$(\Omega' P_{ik,lm})(\Omega' P_{il,km}) \equiv (C'A)(D'B)(cd)^2 \left[\frac{a_i a_m b_i b_m}{(im)^2} - \frac{a_k a_l b_k b_l}{(kl)^2}\right] \equiv 0.$$

The diagonalpoints of the tetragon form a polar-triangle of $(\Omega' X)^2 = 0$.