Mathematics. — *Approximate division of an angle into equal parts.* By J. G. VAN DER CORPUT and H. MOOIJ.

(Communicated at the meeting of March 26, 1949.)

The editors [1]) of Mathematica A published an approximate construction of the trisection, due to M. MARTENS, a former head-master. Although this construction is very accurate for acute angles (with an error smaller than 21′ 24″), it is surpassed in this respect by the very simple construction given by S. C. VAN VEEN [2]), of which the error for acute angles is less than 2′ 36″.

These articles led H. MOOIJ to deal in his thesis [3]) with the problem of dividing a given angle approximately into a number of equal parts and even of the approximate construction of $r\alpha$, where α is a given angle and r denotes a number between zero and one, which can be constructed with a pair of compasses and a ruler.

MOOIJ's construction is based on the following theorem.

First theorem.

Describe a circle, the centre of which coincides with the vertex O of the given angle $AOB = 2\alpha \leqq 180°$.

Suppose that the circle intersects the legs of the angle at A and B.

Let C be the middle of the segment AB. Produce AB to D in such a way, that

$$CD = \frac{r^2+3}{4r} AC.$$

Describe a circle with D as centre and $\frac{3-3r^2}{4r} AC$ as radius, which intersects the smaller arc AB at E. Then $\angle COE$ is approximately equal to $r\alpha$ and the difference is smaller than

$$\tfrac{4}{3} r (1-r^2) (\operatorname{tg} \tfrac{1}{2}\alpha + \tfrac{1}{4}\sin\alpha - \tfrac{3}{4}\alpha).$$

Thus the trisection ($r = \frac{1}{3}$) gives the following construction, which is identical to VAN VEEN's.

Describe a circle with the unit as radius, the centre O of which is the vertex of the given angle $AOB = 2\alpha \leqq 180°$. Assume that the circle intersects the legs of the given angle 2α at A and B. Let C be the middle of AB. Produce AB to D such that

$$CD = \frac{7}{3} AC.$$

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1) Trisectie, Mathematica A **6** (1937—38), p. 1—4.

2) S. C. VAN VEEN, Benaderde trisectie, Mathematica A **7** (1938—39), p. 229—237.

3) H. MOOY, Over de didactiek van de meetkunde benevens benaderingsconstructies ter verdeling van een hoek in gelijke delen, Thesis Amsterdam 1948.

Describe a circle with D as centre and AB as radius, which intersects the smaller arc AB at E. Then $\angle COE$ is approximately equal to $\frac{1}{3}\,\alpha$ and the difference is smaller than

$$\tfrac{32}{81}\,(\mathrm{tg}\,\tfrac{1}{2}\,\alpha + \tfrac{1}{4}\sin\alpha - \tfrac{3}{4}\,\alpha).$$

In this article the proof of the results found by MOOIJ will again be given. Further we shall give a second approximate construction of $r\,\alpha$, which is a little more complicated, but gives a more accurate approximation.
The new construction is based on the following theorem.

Second theorem.

Describe a circle with the unit as radius, the centre O of which coincides with the vertex of the given angle $AOB = 2\alpha \leqq 180°$.
Assume the circle intersects the legs of the given angle at A and B; let C be the middle of AB. Produce AB to D such that

$$CD = \frac{1}{4r}(3 + r^2)AC + \frac{1}{240\,r}(1-r^2)(9-r^2)\,AC^3. \quad . \quad . \quad (1)$$

Describe a circle with D as centre and the radius

$$\frac{3}{4r}(1-r^2)\,AC + \frac{1}{240\,r}(1-r^2)(9-r^2)\,AC^3. \quad . \quad . \quad . \quad . \quad (2)$$

If this circle intersects the smaller are AB of circle O at E, then $\angle COE$ is approximately equal to $r\alpha$ and the difference is smaller than

$$\tfrac{4}{3}\,r\,(\mathrm{tg}\,\tfrac{1}{2}\,\alpha + \tfrac{1}{4}\sin\alpha + \tfrac{3}{80}\sin^3\alpha - \tfrac{3}{80}\,\alpha\sin^2\alpha - \tfrac{3}{4}\,\alpha).$$

As for the trisection ($r = \frac{1}{3}$) we get the following approximate construction.
Describe a circle with the unit as radius, the centre O of which is the vertex of the given angle $OAB = 2\alpha \leqq 180°$.
Assume that the circle intersects the legs of the given angle 2α at A and B and that C is the centre of AB. Produce AB to D, such that

$$CD = \tfrac{7}{3}\,AC + \tfrac{8}{81}\,AC^3.$$

Describe a circle with D as centre and with

$$2\,AC + \tfrac{8}{81}\,AC^3$$

as radius. This circle intersects the smaller arc AB at E. Than $\angle COE$ is approximately equal to [1] α and the difference is smaller than

$$\tfrac{4}{9}\,(\mathrm{tg}\,\tfrac{1}{2}\alpha + \tfrac{1}{4}\sin\alpha + \tfrac{3}{80}\sin^3\alpha - \tfrac{3}{80}\,\alpha\sin^2\alpha - \tfrac{3}{4}\,\alpha).$$

This construction becomes somewhat simpler, if the factor $(9-r^2)$ in the second term of (1) en (2) is replaced by 9.
If $r = \frac{1}{3}$, we get

$$CD = \tfrac{7}{3}\,AC + \tfrac{1}{10}\,AC^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and the radius of the circle with D as centre becomes

$$2\,AC + \tfrac{1}{10}\,AC^3. \qquad \ldots\ldots\ldots \qquad (4)$$

This approximation is extraordinarily accurate. In order to have a survey of the accuracy of the approximation of the trisection, we give the following tables, referring to the three constructions I, II and III; here I denotes VAN VEEN's construction (which also occurs in MOOIJ's thesis); II is the construction according to the formulae (1) en (2) and finally III is the simpler construction in which the formulae (3) and (4) are applied.

Difference from $^1/_3\,\alpha$.

2α	I	II	III
180°	1° 35′ 17″	1° 4′ 9.5″	1° 3′ 7,1″
120°	11′ 19″	4′ 21,2″	4′ 16,9″
90°	2′ 38″	37,3″	35,7″
60°	18″	2,29″	2,25″

For the proof of the above assertions we need the following lemma

Lemma.

Assume the real numbers A, B, C and D satisfy the inequalities

$$A^2 + B^2 \geqq C^2; \quad A^2 + B^2 > D^2; \quad C \neq D$$

and the real number t satisfies

$$A \sin t + B \cos t = D.$$

Then the equation

$$A \sin x + B \cos x = C \qquad \ldots\ldots\ldots \qquad (5)$$

has at least one real solution x for which the inequality

$$|x - t| < \frac{2\,|C - D|}{\sqrt{A^2 + B^2 - C^2} + \sqrt{A^2 + B^2 - D^2}}$$

holds and for which $x - t$ has the same sign as

$$\frac{C - D}{A \cos t - B \sin t}.$$

Moreover

$$A \cos t - B \sin t = \pm \sqrt{A^2 + B^2 - D^2} \neq 0.$$

For the proof we substitute $x = t + u$ in (5) and we introduce z determined by

$$\operatorname{tg} \tfrac{1}{2}\,u = z \qquad (-\pi < u < \pi).$$

Equation (5) becomes

$$A \sin(t + u) + B \cos(t + u) = C,$$

whence

$$A\left(\frac{1-z^2}{1+z^2}\sin t+\frac{2z}{1+z^2}\cos t\right)+B\left(\frac{1-z^2}{1+z^2}\cos t-\frac{2z}{1+z^2}\sin t\right)=C,$$

consequently

$$z^2(A\sin t+B\cos t+C)-2z(A\cos t-B\sin t)-(A\sin t+B\cos t-C)=0,$$

hence

$$(D+C)z^2+2(B\sin t-A\cos t)z+C-D=0. \quad . \quad . \quad (6)$$

The discriminant of this quadratic equation is

$$\Delta=(B\sin t-A\cos t)^2-(C^2-D^2)$$

and therefore

$$\Delta=A^2+B^2-C^2\geqq 0.$$

The roots of (6) are

$$\operatorname{tg}\tfrac{1}{2}u=\frac{A\cos t-B\sin t\pm\sqrt{A^2+B^2-C^2}}{C+D}$$
$$=\frac{A^2+B^2-D^2-A^2-B^2+C^2}{(C+D)(A\cos t-B\sin t\mp\sqrt{A^2+B^2-C^2})}.$$

The sign of $\pm\sqrt{A^2+B^2-C^2}$ can be taken equal to that of $A\cos t-B\sin t$ and then the sign of $\operatorname{tg}\frac{1}{2}u$ is the same as that of

$$\frac{C-D}{A\cos t-B\sin t}.$$

According to our convention u lies between $-\pi$ and π, so that $\frac{1}{2}u$ has the same sign as $\operatorname{tg}\frac{1}{2}u$, thus

$$|\tfrac{1}{2}u|<|\operatorname{tg}\tfrac{1}{2}u|=\frac{|C-D|}{\sqrt{A^2+B^2-C^2}+|A\cos t-B\sin t|}$$
$$=\frac{|C-D|}{\sqrt{A^2+B^2-C^2}+\sqrt{A^2+B^2-D^2}}$$

where $u=x-t$. This establishes the proof.

For the proof of the first theorem we put

$AO=1$; $\angle COE=x$; $CD=p\sin\alpha$; $DE=q\sin\alpha$ and $\angle BDE=y$.

By projecting OED on OC and on AD, we get the relations

$$\cos x-q\sin\alpha\sin y=\cos\alpha$$

and

$$\sin x+q\sin\alpha\cos y=p\sin\alpha.$$

By eliminating y we obtain

$$p\sin\alpha\sin x+\cos\alpha\cos x=1+\tfrac{1}{2}(p^2-q^2-1)\sin^2\alpha,$$

thus

$$A \sin x + B \cos x = C,$$

where

$$A = p \sin \alpha; \quad B = \cos \alpha \text{ and } C = 1 + \tfrac{1}{2}(p^2 - q^2 - 1) \sin^2 \alpha. \quad . \quad (7)$$

We have to determine A and C in such a way, that x is approximately equal to $r\,\alpha$; therefore we put

$$A \sin r\alpha + B \cos r\alpha = D. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Then, according to the above lemma,

$$|r\alpha - x| < \frac{2\,|C-D|}{\sqrt{A^2+B^2-C^2} + \sqrt{A^2+B^2-D^2}} \quad . \quad . \quad . \quad . \quad (9)$$

Expanding $\cos r\,\alpha$ and $\sin r\,\alpha$ in powers of $\sin \alpha$, we get

$$\left.\begin{aligned} \cos r\alpha &= 1 - \frac{r^2}{2!}\sin^2\alpha - \frac{r^2(2^2-r^2)}{4!}\sin^4\alpha - \frac{r^2(2^2-r^2)(4^2-r^2)}{6!}\sin^6\alpha - \ldots \\ \sin r\alpha &= r\sin\alpha + \frac{r(1^2-r^2)}{3!}\sin^3\alpha + \frac{r(1^2-r^2)(3^2-r^2)}{5!}\sin^5\alpha + \ldots \end{aligned}\right\} \quad (10)$$

From the relations (7) and (10) it follows that (8) becomes

$$D = p\sin\alpha \left\{ r\sin\alpha + \frac{r(1^2-r^2)}{3!}\sin^3\alpha + \ldots \right\} + \cos\alpha\cos r\alpha.$$

Differentiation of (10) gives

$$r\cos r\alpha = r\cos\alpha \left\{ 1 + \frac{1^2-r^2}{2!}\sin^2\alpha + \frac{(1^2-r^2)(3^2-r^2)}{4!}\sin^4\alpha + \ldots \right\}.$$

By substituting this result in the above value of D, we get

$$\left.\begin{aligned} D = p\sin\alpha &\left\{ r\sin\alpha + \frac{r(1^2-r^2)}{3!}\sin^3\alpha + \frac{r(1^2-r^2)(3^2-r^2)}{5!}\sin^5\alpha + \ldots \right\} + \\ &+ (1-\sin^2\alpha)\left\{ 1 + \frac{1^2-r^2}{2!}\sin^2\alpha + \frac{(1^2-r^2)(3^2-r^2)}{4!}\sin^4\alpha + \ldots \right\}. \end{aligned}\right\} \quad (11)$$

Putting

$$a_h = \{(2h-1)^2 - \varrho\}\, a_{h-1}; \quad a_0 = 1; \quad \psi(h) = -(2h-1+\varrho) + 2prh$$

and replacing $\sin \alpha$ by s and r^2 by ϱ, we find

$$D = \sum_{h=0}^{\infty} \frac{s^{2h}\, a_{h-2}\{(2h-3)^2-\varrho\}}{(2h)!}\,\psi(h) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

In order to get a good approximation of $r\,\alpha$, we choose p and q in such a way, that the expansion of $C-D$ begins with $\sin^6 \alpha$. This gives

$\tfrac{1}{2}(p^2-q^2-1) = pr - 1 + \tfrac{1}{2}(1-r^2)$ and

$$\frac{pr(1^2-r^2)}{3!} + \frac{(1^2-r^2)(3^2-r^2)}{4!} - \frac{1^2-r^2}{2!} = 0,$$

hence

$$p = \frac{r^2+3}{4r} \text{ and } q = \frac{3-3r^2}{4r}. \quad \ldots \ldots \quad (13)$$

In what follows an upperbound is given for the error, provided that r lies between 0 and 1. Substituting in (12) the value of p given by (13), we get

$$\psi(h) = -2h + 1 - \varrho + \frac{r^2+3}{2r} hr = \tfrac{1}{2}(h-2)(\varrho-1)$$

hence

$$D = \sum_{h=0}^{\infty} \frac{s^{2h}\, a_{h-2}\, \{(2h-3)^2-\varrho\}}{2\,.\,(2h)!} (h-2)(\varrho-1).$$

In $C-D$ the terms with s^0; s^2 and s^4 disappear, hence

$$C-D = -\sum_{h=3}^{\infty} \frac{s^{2h}\, a_{h-2}\, \{(2h-3)^2-\varrho\}}{2\,.\,(2h)!} (h-2)(\varrho-1)$$
$$= (1-\varrho)^2 (3^2-\varrho)\, U s^6,$$

where

$$U = \frac{1}{2\,.\,6!} + \frac{2(5^2-\varrho)\,5^2}{2\,.\,8!} + \frac{3\,.\,(5^2-\varrho)(7^2-\varrho)\,5^4}{2\,.\,10!} + \ldots.$$

By virtue of $0 \leqq \varrho \leqq 1$ the inequality $U \leqq U_0$ is evident, consequently

$$C-D \leqq s^6 (1-\varrho)^2 (3^2-\varrho)\, U_0$$

where U_0 denotes the value of U at the point $\varrho = 0$. Hence

$$C-D \leqq (1-\varrho)^2 (1-\tfrac{1}{9}\varrho)(C-D)_0, \quad \ldots \ldots \quad (14)$$

where $(C-D)_0$ denotes the value which $C-D$ takes for $\varrho = 0$.

The denominator of the right hand side of (9) contains the terms $\sqrt{A^2+B^2-C^2}$ and $\sqrt{A^2+B^2-D^2}$. Here

$$A^2+B^2-C^2 = p^2 s^2 + \cos^2 \alpha - \{1 + \tfrac{1}{2}(p^2-q^2-1)s^2\}^2$$
$$= q^2 s^2 - \tfrac{1}{4}(p^2-q^2-1)^2 s^4.$$

Substitution of the values of p and q, found in (13), gives

$$A^2+B^2-C^2 = \frac{9(1-\varrho)^2}{16\varrho} s^2 - \frac{(1-\varrho)^2}{16} s^4,$$

hence

$$\sqrt{A^2+B^2-C^2} = \frac{3(1-\varrho)}{4r} s \sqrt{1-\frac{\varrho}{9}s^2}. \quad \ldots \quad (15)$$

By putting $t = r\alpha$ in the last formula of our lemma and substituting (7), we obtain

$$\pm\sqrt{A^2+B^2-D^2} = p \sin \alpha \cos r\alpha - \cos \alpha \sin r\alpha \quad \ldots \quad (16)$$

With the aid of the series (10), by substituting $p = \frac{r^2+3}{4r}$, and. putting $c_0 = 1$ and $c_h = (4h^2-\varrho)\, c_{h-1}$ $(h \geqq 1)$, we get

$$\pm \sqrt{A^2+B^2-D^2} = \frac{r(1-\varrho)}{4} \left\{ \frac{3}{\varrho} s - \frac{s^3}{3!} + \frac{2^2-\varrho}{5!} s^5 + \frac{3(4^2-\varrho)(2^2-\varrho)}{7!} s^7 + \ldots \right\}$$

$$= \frac{r(1-\varrho)}{4} \sum_{h=1}^{\infty} \frac{(2h-5)\, c_{h-2}}{(2h-1)!} s^{2h-1}.$$

The right hand side is $\geqq 0$, consequently

$$\sqrt{A^2+B^2-D^2} \geqq \frac{3(1-\varrho)}{4r} s \left(1 - \frac{\varrho}{18} s^2\right)$$

and a fortiori

$$\sqrt{A^2+B^2-D^2} > \frac{3(1-\varrho)}{4r} s \sqrt{1 - \frac{\varrho}{9} s^2} \quad . \quad . \quad . \quad . \quad (17)$$

The relations (9), (14), (15) en (17) give

$$|ra-x| < \frac{2(1-\varrho)^2 (1-\frac{1}{9}\varrho)(C-D)_0}{\frac{3(1-\varrho)s}{4r}\sqrt{1-\frac{1}{9}\varrho s^2} + \frac{3(1-\varrho)s}{4r}\sqrt{1-\frac{1}{9}\varrho s^2}}.$$

From $C_0 = 1 + \frac{1}{4} s^2$ and $D_0 = \frac{3}{4} a \sin a + \cos a$ it follows that

$$(C-D)_0 = 1 + \tfrac{1}{4}\sin^2 a - \tfrac{3}{4} a \sin a - \cos a$$

$$= \sin a \left(\operatorname{tg}\frac{a}{2} + \tfrac{1}{4}\sin a - \tfrac{3}{4} a \right).$$

Hence

$$|ra-x| < \frac{2(1-\varrho)^2(1-\frac{1}{9}\varrho)\, s \left(\operatorname{tg}\frac{a}{2} + \frac{1}{4}\operatorname{tg} a - \frac{3}{4} a\right)}{\frac{2.3(1-\varrho)s}{4r}\sqrt{1-\frac{1}{9}\varrho s^2}}.$$

Consequently

$$|ra-x| < \tfrac{4}{3} r(1-\varrho)(\operatorname{tg}\tfrac{1}{2}a + \tfrac{1}{4}\sin a - \tfrac{3}{4}a).$$

This establishes the proof of the first theorem. For the proof of the second theorem we put

$$p = \frac{P}{r} + \frac{Q}{r} s^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Then

$$C = 1 + \tfrac{1}{2}(p^2-q^2-1)s^2 = 1 + K s^2 + L s^4, \quad . \quad . \quad . \quad (19)$$

where P, Q, K and L are properly chosen functions of r. Formula (8) gives

$$D = \left(\frac{P}{r} + \frac{Q}{r} s^2\right) s \sin ra + \cos a \cos ra,$$

hence

$$D=(Ps^2+Qs^4)\sum_{h=0}^{\infty}\frac{a_h\,s^{2h}}{(2h+1)!}+(1-s^2)\sum_{h=0}^{\infty}\frac{a_h}{(2h)!}\,s^{2h},$$

where

$a_h=(1^2-\varrho)(3^2-\varrho)\ldots((2h-1)^2-\varrho)$ and $a_0=1$, so $a_h=\{(2h-1)^2-\varrho\}\,a_{h-1}$, consequently

$$D=\sum_0^{\infty}s^{2h}\left\{\frac{a_h}{(2h)!}-\frac{a_{h-1}}{(2h-2)!}+\frac{P\,a_{h-1}}{(2h-1)!}+\frac{Q\,a_{h-2}}{(2h-3)!}\right\}$$

$$=\sum_0^{\infty}s^{2h}\,a_{h-2}\left\{\frac{(2h-1)^2-\varrho}{(2h)!}((2h-3)^2-\varrho)-\frac{(2h-3)^2-\varrho}{(2h-2)!}+\right.$$

$$\left.+\frac{(2h-3)^2-\varrho}{(2h-1)!}\,P+\frac{Q}{(2h-3)!}\right\}.$$

This gives

$$D=\sum_0^{\infty}\frac{s^{2h}\,a_{h-2}}{(2h)!}\,\varphi(h)$$

where

$$\varphi(h)=-\{(2h-1)+\varrho\}\{(2h-3)^2-\varrho\}+2h\{(2h-3)^2-\varrho\}P+$$
$$+2h(2h-1)(2h-2)Q.$$

The functions P, Q, K and L are chosen such that the expansion of $C-D$ begins with s^8. By virtue of

$$D=1+\frac{s^2}{2!}(-1-\varrho+2P)+\frac{s^4}{4!}\{-(3+\varrho)(1-\varrho)+4(1-\varrho)P+4.3.2.Q\}+$$
$$+\frac{s^6(1-\varrho)}{6!}\{-(5+\varrho)(9-\varrho)+6(9-\varrho)P+6.5.4Q\}+\ldots$$

we get in connection with (19)

$$2K=-1-\varrho+2P;\quad\ldots\ldots\ldots\quad(20)$$

$$24L=-(3+\varrho)(1-\varrho)+4(1-\varrho)P+4.3.2.Q;\quad\ldots\quad(21)$$

$$-(5+\varrho)(9-\varrho)+6(9-\varrho)P+6.5.4.Q=0.\quad\ldots\quad(22)$$

Substitution in (19) of the values of p and $2K$, found respectively in (18) and (20) furnishes

$$q^2\varrho=(P-\varrho)^2+2s^2(PQ-L\varrho)+s^4Q^2.\quad\ldots\ldots\quad(23)$$

To simplify we put the right hand side of this equation equal to a perfect square, by choosing

$$PQ-L\varrho=(P-\varrho)Q,\ \text{hence}\ L=Q.$$

From (21) follows

$$4(1-\varrho)P=(3+\varrho)(1-\varrho),\ \text{hence}\ P=\frac{3+\varrho}{4}.\quad\ldots\ldots\quad(24)$$

Formula (22) gives

$$120\,Q = (5+\varrho)(9-\varrho) - 6\,(9-\varrho)\,.\,\frac{3+\varrho}{4},$$

hence

$$Q = L = \frac{(1-\varrho)(9-\varrho)}{240}.$$

By (20) and (24) we obtain

$$K = \frac{1-\varrho}{4}.$$

Now (23) becomes

$$\varrho\, q^2 = (P-\varrho+Q s^2)^2 \text{ and } r\,q = \pm\,(P-\varrho+Q s^2).$$

The left hand side of the last relation and also both $P-\varrho$ and $Q\,s^2$ are positive, so that the plus sign holds good. This gives

$$r\,q = \frac{1-\varrho}{240}\,\{180 + (9-\varrho)\,s^2\}$$

and by (18)

$$r\,p = \frac{3+\varrho}{4} + \frac{(1-\varrho)(9-\varrho)}{240}\,s^2 . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

In order to obtain an upper bound of the error, we deduce from our lemma that this error is

$$|\,r\,a - x\,| < \frac{2\,|\,C-D\,|}{\sqrt{A^2+B^2-C^2} + \sqrt{A^2+B^2-D^2}};$$

here

$$C-D = -\sum_{h=4}^{\infty} \frac{s^{2h}\,a_{h-2}}{(2h)!}\,\varphi(h)$$

and

$$\varphi(h) = -\{2h-1+\varrho\}\,\{(2h-3)^2-\varrho\} + \\ + 2h\,\{(2h-3)^2-\varrho\}\,P + 2h\,(2h-1)\,(2h-2)\,Q.$$

Consequently

$$\varphi(h) = -\,\{(2h-1)+\varrho\}\,\{(2h-3)^2-\varrho\} + \\ + 2h\,\{(2h-3)^2-\varrho\}\,\frac{3+\varrho}{4} + 2h\,(2h-1)\,(2h-2)\,\frac{(1-\varrho)(9-\varrho)}{240},$$

which gives after reduction

$$\varphi(h) = \frac{(1-\varrho)(h-3)}{60}\,\{-102\,h^2 + 267\,h - 180 - \varrho\,(2\,h^2+3\,h-20)\},$$

hence

$$C-D = \frac{1-\varrho}{60}\sum_{h=4}^{\infty}(h-3)\,\frac{s^{2h}\,a_{h-2}}{(2h)!}\,\{102\,h^2 - 267\,h + 180 + \varrho\,(2\,h^2+3h-20)\}.$$

In this relation we have for $h \geqq 4$

$$a_{h-2} = (1-\varrho)(1-\tfrac{1}{9}\varrho)\, b_{h-2}(\varrho),$$

where $b_{h-2}(\varrho)$ is a polynomial in ϱ, which has in the interval $0 \leqq \varrho \leqq 1$ its maximum value at the point $\varrho = 0$, therefore

$$0 \leqq a_{h-2} \leqq (1-\varrho)(1-\tfrac{1}{9}\varrho)\, b_{h-2}(0) = (1-\varrho)(1-\tfrac{1}{9}\varrho)\, a_{h-2}(0).$$

Further

$$102\,h^2-267\,h+180+\varrho\,(2h^2+3h-20) \leqq (1+\tfrac{1}{31}\varrho)(102\,h^2-267\,h+180);$$

for this relation is equivalent to the inequality

$$62\,h^2+93\,h-620 \leqq 102\,h^2-267\,h+180,$$

which is evident in virtue of $h^2-9\,h+20 \geqq 0$. Hence

$$C-D \leqq$$

$$\leqq \frac{(1-\varrho)^2}{60}(1-\tfrac{1}{9}\varrho)\sum_{h=4}^{\infty}(h-3)\frac{s^{2h}}{(2h)!}\, a_{h-2}(0)\,(1+\tfrac{1}{31}\varrho)\,(102\,h^2-267\,h+180).$$

This furnishes

$$C-D \leqq (1-\varrho)(1-\tfrac{1}{9}\varrho)(1+\tfrac{1}{31}\varrho)(C-D)_0. \quad . \quad . \quad . \quad (26)$$

From (7), (8) and (25) we deduce

$$D = \frac{3+\varrho}{4r}\sin a \sin ra + \frac{(1-\varrho)(9-\varrho)}{240\,r}\sin^3 a \sin ra + \cos a \cos ra,$$

hence

$$D_0 = \tfrac{3}{4}\,a \sin a + \tfrac{3}{80}\,a \sin^3 a + \cos a.$$

Further

$$C_0 = 1 + \tfrac{1}{4}s^2 + \tfrac{3}{80}s^4,$$

hence

$$(C-D)_0 = 1 + \tfrac{1}{4}s^2 + \tfrac{3}{80}s^4 - \tfrac{3}{4}\,as - \tfrac{3}{80}\,as^3 - \cos a,$$

consequently

$$(C-D)_0 = \sin a\,(\mathrm{tg}\,\tfrac{1}{2}a + \tfrac{1}{4}\sin a + \tfrac{3}{80}\sin^3 a - \tfrac{3}{80}\,a\sin^2 a - \tfrac{3}{4}\,a). \quad (27)$$

Further

$$A^2+B^2-C^2 = p^2\sin^2 a + \cos^2 a - (1 + K\sin^2 a + L\sin^4 a)^2$$

$$= \left(\frac{P}{r}+\frac{Qs^2}{r}\right)^2 s^2 + 1 - s^2 - (1 + K^2s^4 + L^2s^8 + 2Ks^2 + 2Ls^4 + 2KLs^6),$$

where $Q = L$, hence

$$A^2+B^2-C^2 = \frac{P^2}{\varrho}s^2 + \frac{2PQ}{\varrho}s^4 + \frac{Q^2}{\varrho}s^6 +$$

$$+ 1 - s^2 - 1 - K^2s^4 - Q^2s^8 - 2Ks^2 - 2Qs^4 - 2KQs^6$$

$$= s^2\left\{\frac{P^2}{\varrho} - 1 - 2K + s^2\left(\frac{2PQ}{\varrho} - K^2 - 2Q\right) + s^4\left(\frac{Q^2}{\varrho} - 2QK\right) - Q^2s^6\right\}.$$

Substitution of the values of P, Q and K gives

$$\sqrt{A^2+B^2-C^2}=\frac{1-\varrho}{4}\,s\sqrt{\left\{\frac{9}{\varrho}+\frac{9-11\varrho}{10\varrho}s^2+\frac{(9-\varrho)(9-121\varrho)}{3600\varrho}s^4-\frac{(9-\varrho)^2}{3600}s^6\right\}}.$$

In order to deduce from this relation the inequality

$$\sqrt{A^2+B^2-C^2}\geqq\frac{1-\varrho}{4}\frac{3s}{\sqrt{\varrho}}(1-\tfrac{1}{9}\varrho)(1+\tfrac{1}{31}\varrho), \quad . \quad . \quad . \quad (28)$$

we remark that this inequality is equivalent to

$$\frac{9}{\varrho}+\frac{9-11\varrho}{10\varrho}\cdot s^2+\frac{(9-\varrho)(9-121\varrho)}{3600\varrho}s^4-\frac{(9-\varrho)^2}{3600}s^6 \geqq\frac{9}{\varrho}(1-\tfrac{2}{9}\varrho+\tfrac{1}{81}\varrho^2)\left(1+\frac{2}{31}\varrho+\frac{1}{31^2}\varrho^2\right)$$

and therefore equivalent to

$$\left(\frac{81}{3600}s^4+\frac{9}{10}s^2\right)\frac{1}{\varrho}+\left(\frac{44}{31}-\frac{11}{10}s^2-\frac{1098}{3600}s^4-\frac{81}{3600}s^6\right)$$
$$+\left(\frac{121}{3600}s^4+\frac{18}{3600}s^6+\frac{74}{9.961}\right)\varrho-\left(\frac{1}{3600}s^6+\frac{44}{9.961}\right)\varrho^2-\frac{1}{9.961}\varrho^3\geqq 0.$$

To prove this inequality it is sufficient to consider the most unfavourable case, viz. $s=1$; in this case the inequality becomes

$$\frac{369}{400}\frac{1}{\varrho}-\frac{101}{12400}+\frac{163179}{961.3600}\varrho-\frac{18561}{961.3600}\varrho^2-\frac{1}{9.961}\varrho^3\geqq 0,$$

which is true for every value of ϱ between 0 and 1.

By substituting in (16) the value of p found in (25) we get

$$\sqrt{A^2+B^2-D^2}=\frac{3(1-\varrho)}{4r}s+\frac{(9-11\varrho)(1-\varrho)}{240r}s^3-\frac{r(1-\varrho)}{4\,.\,4!}s^5\ldots$$
$$+\sum_{n=4}^{\infty}\frac{r(2^2-\varrho)\ldots((2n-6)^2-\varrho)(1-\varrho)(2n-7)\{102n^2-369n+339+(n-3)(2n+7)\varrho\}s^{2n-1}}{120(2n-1)!}.$$

This gives

$$\left.\begin{aligned}\sqrt{A^2+B^2-D^2}\geqq\frac{3(1-\varrho)}{4r}s+\frac{(9-11\varrho)(1-\varrho)}{240r}s^3-\\ -\frac{r(1-\varrho)}{4\,.\,4!}s^5\geqq\frac{1-\varrho}{4}\frac{3s}{\sqrt{\varrho}}(1-\tfrac{1}{9}\varrho)(1+\tfrac{1}{31}\varrho).\end{aligned}\right\}\ . \quad (29)$$

In fact this relation becomes after reduction

$$\tfrac{1}{20}s^2+\tfrac{22}{279}\varrho-\tfrac{11}{180}\varrho s^2+\tfrac{1}{279}\varrho^2-\tfrac{1}{72}\varrho s^4\geqq 0,$$

that is

$$\tfrac{1}{20} s^2 + \varrho\left(\tfrac{22}{279} - \tfrac{11}{180} s^2 - \tfrac{1}{72} s^4\right) + \tfrac{1}{279} \varrho^2 \geqq 0$$

and this inequality holds because for $0 \leqq s \leqq 1$ the factor of ϱ is at least equal to

$$\tfrac{22}{279} - \tfrac{11}{180} - \tfrac{1}{72} > 0.$$

Now by virtue of (26), (27), (28) and (29) the relation (9) becomes

$$|ra - x| <$$

$$< \frac{2(1-\varrho)(1-\frac{1}{9}\varrho)(1+\frac{1}{31}\varrho)\sin a\,(\operatorname{tg}\frac{1}{2}a + \frac{1}{4}\sin a + \frac{3}{80}\sin^3 a - \frac{3}{80}a\sin^2 a - \frac{3}{4}a)}{2\,\frac{1-\varrho}{4}\,\frac{3s}{\sqrt{\varrho}}\,(1-\frac{1}{9}\varrho)(1+\frac{1}{31}\varrho)}$$

which gives after reduction

$$|ra - x| < \tfrac{4}{3} r\,(\operatorname{tg}\tfrac{1}{2}a + \tfrac{1}{4}\sin a + \tfrac{3}{80}\sin^3 a - \tfrac{3}{80} a \sin^2 a - \tfrac{3}{4} a).$$

This becomes in the special case $r = \frac{1}{3}$

$$|ra - x| < \tfrac{4}{9}\,(\operatorname{tg}\tfrac{1}{2}a + \tfrac{1}{4}\sin a + \tfrac{3}{80}\sin^3 a - \tfrac{3}{80} a \sin^2 a - \tfrac{3}{4} a).$$

This establishes the proof of the second theorem.