Mathematics. — SOMMERFELD's *Polynomial Method in the Quantum Theory*. By A. RUBINOWICZ. (Communicated by Prof. H. A. KRAMERS.)

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§ 1. *Scope of the paper.* A. SOMMERFELD [1]) has devised a method which enables us to treat many of the most simple but also most important eigenvalue problems in a very easy manner. His polynomial method starts with a splitting of the eigenfunction f of the originally given differential equation into two factors E and P

$$f = EP. \qquad (1)$$

The factor E takes care of the convergence of the normalization integral and of the fulfilment of the boundary conditions. The factor P however is assumed to be a polynomial, so that it does not disturb these properties of E. We suppose moreover that P is a solution of a differential equation of the second order with a recurrence formula containing only two terms, that is one of the form

$$(A_2 + B_2 x^h)\, x^2 \frac{d^2 P}{dx^2} + 2\,(A_1 + B_1 x^h)\, x \frac{dP}{dx} + (A_0 + B_0 x^h)\, P = 0. \qquad (2)$$

A_i and B_i [2]) are constants and h is a positive integer.

In the present paper we will indicate the eigenvalue problems which can be treated by SOMMERFELDs polynomial method. In § 2 we suppose that the polynomial P in (1) is a solution of (2). Then a relation connecting the coefficients of the differential equation (2) and the original one determines the general form of the potential V which appears in the original differential equation. But the function V obtained in such a way contains the eigenvalue parameter λ and the coefficients A_i and B_i which depend generally on λ. If V has a real physical meaning it can however not depend on λ. That means that there are some relations between λ and the coefficients A_i and B_i. Further relations follow if we require that E guarantees the fulfilment of boundary conditions. All these equations determine not only uniquely the potential function V but settle also the eigenvalues.

From this point of view we treat in the following sections some special problems completely. In § 3 we deal with the radial functions of the spherical symmetric case. In § 4 we discuss the differential equation of associated Legendre functions. In § 5 we start from the same differential equation as in § 4, but apply the polynomial method after a linear transformation of the independent variable. In this way we obtain the differential equation of JACOBI polynomials.

§ 2. *The connection between the original and the polynomial differential equations.* Going over from the "original" differential equation

$$f'' + 2af' + bf = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

with the aid of (1.1) [3]) to the differential equation of the polynomials

$$P'' + 2\alpha P' + \beta P = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

we get between the coefficients of both the differential equations the relations

$$\alpha = \frac{E'}{E} + a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

$$\beta = \frac{E''}{E} + 2a\frac{E'}{E} + b. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3b)$$

Eliminating E we obtain a relation between the coefficients of both the differential equations (1) and (2)

$$a^2 + a' - b = \alpha^2 + \alpha' - \beta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

We denote this expression in the following considerations by S.

We assume that the coefficients of (1) are real numbers and that (1) is the differential equation of an eigenvalue problem. (1) is then selfadjoint and has the form

$$\frac{d}{dx}\left(p\frac{df}{dx}\right) - qf + \lambda\varrho f = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

λ denotes the eigenvalue parameter and $\varrho(x)$ the density function. It appears in the integral

$$\int_{x_1}^{x_2} f_2^* f_1 \varrho \, dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which decides in case of discrete eigenvalues about normalization and orthogonality of eigenfunctions. x_1 and x_2 are the boundaries of the fundamental interval.

Comparing (1) and (5) we obtain

$$a = \frac{1}{2}\frac{p'}{p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7a)$$

$$b = -\frac{q - \lambda\varrho}{p}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7b)$$

According to (4) and (7) we can represent S in the form

$$S = \frac{1}{p}\left(\frac{1}{2}p'' - \frac{1}{4}\frac{p'^2}{p} + q - \lambda\varrho\right). \quad . \quad . \quad . \quad . \quad . \quad (8)$$

so that q becomes

$$q = pS + \lambda\varrho + \frac{1}{4}\frac{p'^2}{p} - \frac{1}{2}p''. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The form of the expression S which appears in (9) is known because we suppose that the differential equation for P is given by (1.2) and therefore

$$\alpha = \frac{1}{x}\frac{A_1 + B_1 x^h}{A_2 + B_2 x^h} \qquad \beta = \frac{1}{x^2}\frac{A_0 + B_0 x^h}{A_2 + B_2 x^h} \quad . \; . \; . \quad (10)$$

so that according to (4)

$$\left.\begin{aligned} S = \alpha^2 + \alpha' - \beta = \frac{(A_1 + B_1 x^h)\,[A_1 - A_2 + (B_1 - (h+1)\, B_2, x^h]}{x^2\,(A_2 + B_2 x^h)^2} - \\ - \frac{A_0 + (B_0 - h\, B_1)\, x^h}{x^2\,(A_2 + B_2 x^h)}. \end{aligned}\right\} \quad (11)$$

Our final result is: If the differential equation (5) with given p and q is solvable by $f = E\,P, P$ being a solution of (1.2), q must be of the form (9) where S has the form (11).

We apply this proposition in cases where we can split the SCHROEDINGER equation

$$\triangle\, \psi + \varkappa\, (\mathcal{E} - V)\, \psi = 0. \qquad \varkappa = \frac{2\,m}{\hbar^2}. \quad . \; . \; . \; . \quad (12)$$

into a number of differential equations of the form (5). Both p and ϱ are then completely determined by the coordinates used for the separation of the variables and q contains generally an expression given by the potential function V. By (9) and (11) are settled the forms of the q's of all these differential equations therefore also the form of the potential function V.

A more exact determination of V we obtain by the demand that the coefficients of V can not depend on the eigenvalue parameter λ, if V has a real physical meaning. This takes place if the expressions (9) for q do not depend on λ [4]). But this can be fulfiled only if the coefficients B_i which appear in S are functions of λ. To find out this dependence we can expand the expression

$$p\,S + \lambda\,\varrho \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

contained in (9) in a power series in powers of $x—x_0$ (x_0 arbitrary) eventually after multiplication with a function of x. On this occasion we also find, that only for distinct values of h the expression (13) can be made independent of λ.

Further conclusions as to the A_i and B_i and so as to the potential V we can draw from the boundary conditions for the eigenfunctions. As a rule the fundamental interval in the quantum theory in bounded by two singular points of the equation (5). In such points the solution has a tendency towards becoming infinite. From the mathematical point of view it is the task of the boundary conditions to suppress this tendency.

If we confine ourselves to the discrete eigenvalue spectrum we must claim that the integral (6) is convergent for any two eigenfunctions f_1 and f_2, since otherwise we can not speak of their normalization or orthogonality.

Also in case of singular boundary points x_1 and x_2 we prove the orthogonality of two eigenfunctions f_1 and f_2 with the eigenvalues λ_1 and λ_2 by the aid of the well known relation

$$(\lambda_2 - \lambda_1) \int_{x_1}^{x_2} f_2^* f_1 \varrho \, dx = \lim_{x \to x_2} \Phi(x) - \lim_{x \to x_1} \Phi(x) \quad . \quad . \quad . \quad (14)$$

where

$$\Phi(x) = p(x) \left[f_2^*(x) \frac{df_1(x)}{dx} - f_1(x) \frac{df_2^*(x)}{dx} \right]. \quad . \quad . \quad . \quad (15)$$

To apply (14) in this case in the usual manner we have to claim not only the convergence of the integral (6) appearing in (14) but also the existence and equality of both the limits

$$\lim_{x \to x_1} \Phi(x) = \lim_{x \to x_2} \Phi(x). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Physical arguments can stipulate to claim more, e.g. that (16) is zero or the eigenfunctions are periodic. But also mathematical motives can do it if e.g. the eigenfunctions have to form a complete set of orthogonal functions.

Both factors E and P which form f can facilitate or make more difficult the fulfilment of the boundary conditions. P is a product of x^σ with a "genuine" polynomial, beginning with a constant and consisting of integer powers of x^h. The exponent σ is given here by the determining fundamental equation

$$\sigma(\sigma - 1) A_2 + 2\sigma A_1 + A_0 = 0. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

But also the form of E is determined. According to (3a) we get

$$E = \exp\left(\int (\alpha - a) \, dx\right).$$

From (7a) we find

$$\exp - \int a \, dx = \frac{\text{const}}{p^{1/2}}$$

whereas $\exp \int \alpha \, dx$ may be evaluated by the use of (10). Carrying out this calculation we have to distinguish three cases, according to the disappereance or non-disappereance of the constants A_2 and B_2. We get without an insignificant constant

(I) in case of $A_2 \neq 0$, $B_2 \neq 0$

$$E = p^{-1/2} x^{A_1/A_2} (A_2 + B_2 x^h)^{\frac{1}{h}\left(\frac{B_1}{B_2} - \frac{A_1}{A_2}\right)} \quad . \quad . \quad . \quad . \quad (18a)$$

(II) in case of $A_2 = 0$, $B_2 \neq 0$

$$E = p^{-1/2} x^{B_1/B_2} e^{-\frac{1}{h}\frac{A_1}{B_2}\frac{1}{x^h}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (18b)$$

(III) in case of $A_2 \neq 0$, $B_2 = 0$

$$E = p^{-1/2} x^{A_1/A_2} e^{\frac{1}{h}\frac{B_1}{A_2} x^h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18c)$$

The role played by the singular and zero points of ϱ, p, P and E in the fulfilment of the boundary conditions we shall discuss when considering the different special cases.

To these relations we have to add SOMMERFELDs condition of breaking off the power series

$$(\sigma+n)(\sigma+n-1)B_2+2(\sigma+n)B_1+B_0=0 \quad . \quad . \quad . \quad (19)$$

which makes of P a polynomial and determines the eigenvalues of λ in their dependence of an integer n, divisible by h.

Finally we indicate a very useful property of the polynomial equation (2). It does not change its form, if we multiply the polynomial P with a given power of x. Putting $E=x^\nu$ we get from (3)

$$a=\alpha-\frac{\nu}{x}, \quad b=\beta-2\alpha\frac{\nu}{x}+\frac{\nu(\nu+1)}{x^2}.$$

We obtain therefore for $f=x^\nu P$ the differential equation (1), in which the coefficients a, b are of the form (10) of the coefficients α, β.

In both the cases (I) and (III) in which $A_2 \neq 0$ we shall use the abbreviations

$$a_i=\frac{A_i}{A_2}, \quad b_i=\frac{B_i}{A_2}.$$

§ 3. *Spheric symmetric field of force.* Splitting off from a Schrödinger eigenfunction a spherical harmonic we obtain for the radial function $R(x)$ the differential equation

$$\frac{d}{dx}x^2\frac{dR}{dx}+\varkappa\left[\mathcal{E}x^2-Vx^2-\frac{1}{\varkappa}l(l+1)\right]R=0 \quad . \quad . \quad . \quad (1)$$

i.e. the differential equation (2.5) with

$$p=\varrho=x^2, \ \lambda=\varkappa\mathcal{E}, \ q=\varkappa V x^2+l(l+1) \quad . \quad . \quad . \quad . \quad (2)$$

V is here the potential function and l the azimuthal quantum number.

From (2) and (2.11) we obtain V in the form

$$V=\frac{S}{\varkappa}-\frac{l(l+1)}{\varkappa x^2}+\mathcal{E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We assume that the fundamental interval is given by $0<x<+\infty$ and use for our considerations the factor E first.

To make the normalizing integral (cp. (2.6))

$$\int_0^\infty x^2 R^* R\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

convergent at its upper limit we have to assume case III i.e.

$$A_2 \neq 0, \quad B_2=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

According to (2.18c) and (2) E becomes

$$E=x^{\frac{A_1}{A_2}-1}e^{\frac{1}{h}\frac{B_1}{A_2}x^h} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and the convergence for $x \to +\infty$ requires $B_1/A_2 < 0$. Finally we assume

$$A_1 = A_2 \quad \ldots \ldots \ldots \ldots \quad (7)$$

to unite the x-power from E with the polynomial P.

For further considerations we use V. According to (2. 11), (5) and (7) the expression S is given here by

$$S = \frac{1}{x^2}[-a_0 + ((h+1)\,b_1 - b_0)\,x^h + b_1^2\,x^{2h}].$$

To free V, Eq. (3), from $\mathcal{E}$ we have here only both the possibilities $h = 1$ or $= 2$. For $h = 1$ the potential becomes

$$V = \frac{c_{-2}}{x^2} + \frac{c_{-1}}{x} + c_0, \quad \ldots \ldots \ldots \quad (8)$$

where the constants

$$c_{-2} = -\frac{1}{\varkappa}(a_0 + l(l+1)), \quad \ldots \ldots \quad (9a)$$

$$c_{-1} = \frac{1}{\varkappa}(2\,b_1 - b_0) \quad \ldots \quad (9b), \qquad c_0 = \mathcal{E} + \frac{b_1^2}{\varkappa} \quad \ldots \quad (9c)$$

are independent of $\mathcal{E}$. Therefore we obtain for V a Coulomb potential superposed by a potential inversely proportional to the square of the distance. The coefficients c_i in (8) are arbitrary because their dependence on A_i, B_i does not imply any connection between them.

To the potential (8) belongs the RYDBERG formula. From (2. 19) and (5) follows

$$b_0 = -2\,b_1\,(n + \sigma) \,. \quad \ldots \ldots \ldots \quad (10)$$

and hence, in accordance with (9b), $c_{-1} = \frac{b_1}{2}\,(n + \sigma + 1)$ so that we obtain from (9c) in fact the RYDBERG formula

$$\mathcal{E} = -\frac{\varkappa}{4}\,\frac{(c_{-1})^2}{(n + \sigma + 1)^2} + c_0. \quad \ldots \ldots \ldots \quad (11)$$

Supposing further, that the Coulomb potential has the right constant $c_{-1} = -e^2 Z$ we obtain in (11) the RYDBERG constant. Eq. (9a) not yet used determines σ and hence the RYDBERG correction. From (2. 17), (7) and (9a) we obtain

$$\sigma(\sigma + 1) = -a_0 = l(l+1) + \varkappa c_{-2}. \quad \ldots \ldots \quad (12)$$

In a pure Coulomb field, there is $c_{-2} = 0$ and therefore $\sigma = l$ (for $\sigma = -l - 1$ the normalizing integral (4) is not convergent) so that we obtain the BALMER formula.

In case $h = 2$ we have

$$V = \frac{c_{-2}}{x^2} + c_2\,x^2 + c_0$$

where

$$c_{-2} = -\frac{1}{\varkappa}(a_0 + l(l+1)) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13a)$$

$$c_2 = \frac{1}{\varkappa} b_1^2 \quad . \quad . \quad . \quad (13b), \qquad c_0 = \mathcal{E} + \frac{1}{\varkappa}(3b_1 - b_0) \quad . \quad . \quad (13c)$$

The potential V is consequently given by a superposition of an elastic potential and of a potential inversely proportional to the square of the distance.

The dependence of $\mathcal{E}$ upon the quantum numbers we obtain from (10), (13b) and (13c)

$$\mathcal{E} = -2\frac{b_1}{\varkappa}(n+\sigma+\tfrac{3}{2}) + c_0 = 2\sqrt{\frac{c_2}{\varkappa}}(n+\sigma+\tfrac{3}{2}) + c_0 . \quad . \quad (14)$$

The positive sign of the square root is determined by (6).

If to the pure elastic field of force corresponds a frequency ω (in 2π sec), we have to put $c_2 = \frac{m}{2}\omega^2$. Like in case $h = 1$ the constant σ is given by (12).

In case of a pure elastic potential there is $c_{-2} = 0$ and therefore $\sigma = l$. For the eigenvalues of the spatial harmonic oscillator we obtain then

$$\mathcal{E} = (n + l + \tfrac{3}{2})\,\hbar\,\omega + c_0 . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The general case (14) we can conceive now as (15) with a RYDBERG correction σ. The constant c_0 is in all the formulae of Legendre functions arbitrary and we can put $c_0 = 0$, if V is normalized as usual.

§ 4. *The differential equation for associated Legendre functions.* To have an example of an eigenvalue problem in a finite fundamental interval we generalize the equation for associated Legendre functions

$$\frac{d}{dx}\left((1-x^2)\frac{dK}{dx}\right) + \left(\lambda - \frac{m^2}{1-x^2}\right)K = 0$$

putting $V(x)$ for $m^2/(1-x^2)$

$$\frac{d}{dx}\left((1-x^2)\frac{dK}{dx}\right) + (\lambda - V(x))\,K = 0 . \quad . \quad . \quad . \quad . \quad (1)$$

We obtain hence the differential equation (2.5) with

$$p = 1 - x^2, \quad \varrho = 1, \quad q = V . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (2.9) and (2) we get therefore

$$V = \frac{1}{1-x^2} + S(1-x^2) + \lambda . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For the fundamental interval we choose $-1 < x < +1$ and use first for our considerations the factor E, given by one of the Eqs. (2.18).

The zero points of p, which according to (2) are situated in $x = \pm 1$, endanger the convergence of the normalizing integral given according to (2. 6) and (2) by

$$\int_{-1}^{+1} K^* K \, dx \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

because $p^{-\frac{1}{2}}$ appears in the factors E in all the cases (2. 18). This danger can be eliminated only in case I, Eq. (2. 18*a*), where

$$A_2 \neq 0, \quad B_2 \neq 0 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5)$$

and where according to (2. 18*a*) and (2)

$$E = x^{\frac{A_1}{A_2}} \frac{(A_2 + B_2 x^h)^{\frac{1}{h}\left(\frac{B_1}{B_2} - \frac{A_1}{A_2}\right)}}{(1 - x^2)^{1/2}}.$$

To avoid E disturbing the convergence of the normalization integral (4) in the endpoints $x = \pm 1$ of the fundamental interval and to secure the finiteness of the expressions (2. 15) the binomial $A_2 + B_2 x^h$ must be divisible by $1 - x^2$. Hence follows that

$$A_2 = -B_2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6)$$

and

$$h = \text{even integer.} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7)$$

Both our demands restrict also the variability of the exponent of $A_2 + B_2 x^h$; the demand that the expressions (2. 15) are finite goes further and claims in accordance with (6) that

$$\frac{A_1 + B_1}{A_2} + 1 \leqq 0. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

To transpose x^{A_1/A_2} from E to P we must put

$$A_1 = 0. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (9)$$

Now let us use V. In accordance with (5), (6) and (9) we obtain from (2. 11)

$$S = x^{h-2} b_1 \frac{-1 + (b_1 + h + 1) x^h}{(1 - x^h)^2} - \frac{a_0 + (b_0 - h b_1) x^h}{x^2 (1 - x^h)}. \quad . \quad (10)$$

Developing S in a power series we get from the expression $S p$ appearing in Eq. (3)

$$\left.\begin{aligned} Sp = S(1 - x^2) = -\frac{a_0}{x^2} + a_0 - M(x^{h-2} - x^h) - (M+N)(x^{2h-2} - x^{2h}) + \\ + (M + 2N)(x^{3h-2} - x^{3h}) + \ldots \end{aligned}\right\} \quad (11)$$

where $M = a_0 + b_0 - (h - 1) b_1$, $N = -b_1 (b_1 + h)$.

If V is independent of λ the coefficients of a development of V in a power series can also not depend on λ. Hence in accordance with (3) the coefficients of a power series development of

$$S(1 - x^2) + \lambda \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

must be independent of λ.

First we assume that $h \neq 2$, i.e. according to (7): $h = 4, 6, 8, \ldots.$ In accordance with (11) and (12) then the values of a_0, $a_0 + \lambda$, M, N must be constant. But from the simultaneous constancy of a_0 and $a_0 + \lambda$ it follows that also the eigenvalue parameter λ is invariable. In such a case however our problem is not an eigenvalue problem.

For $h = 2$ we have to demand, that only the three quantities a_0, $a_0 - M + \lambda$ and N are constant. From this and the Eqs. (2.17) and (2.19) we could determine the dependence of the eigenvalues λ upon the quantum numbers. But we may come to this conclusion in an easier way.

For $h = 2$ we obtain from (3) and (10) for V the expression

$$V = \frac{A}{1-x^2} + \frac{B}{x^2} + C \quad \ldots \ldots \ldots \quad (13)$$

where the constants are given by

$$A = (b_1 + 1)^2 \text{ (14a)}, \quad B = -a_0 \text{ (14b)}, \quad C = \lambda - (b_1^2 + b_1 + b_0). \quad \text{(14c)}$$

If we put $b_1 + 1 = -m$ and remark that from (2.18) it follows $b_0 = (\sigma + n)(\sigma + n - 1) - 2b_1(\sigma + n)$ we finally obtain from (14c)

$$\lambda = (\sigma + m + n)(\sigma + m + n + 1) + C. \quad \ldots \quad (15)$$

In accordance with (8) and (9), m is here a positive constant.

From (2.17) and (9) we obtain for σ the relation

$$\sigma(\sigma - 1) = -a_0 = B. \quad \ldots \ldots \ldots \quad (16)$$

By SOMMERFELD's polynomial method we can therefore solve a slightly more general differential equation than the one for the associated Legendre functions. This last differential equation we obtain by putting $B = 0$. In this case is $\sigma = 0$ or $= 1$. Remembering that n is an even integer ($h = 2!$) and hence $n + \sigma$ an arbitrary positive integer we see that (15) represents the well known eigenvalues of the differential equation for the associated Legendre functions.

To demonstrate by an example the simplifications caused by the transposition of the x-power from E to P, we indicate the results arrived at without the supposition (8). Instead of (11) we obtain an expression in which is substituted

$$a_0 - a_1(a_1 - 1) \text{ for } a_0, \quad M - a_1(2a_1 + 2b_1 + h - 1) \text{ for } M,$$
$$N - a_1(a_1 + 2b_1 + h) \text{ for } N.$$

But this does not alter the conclusion that $h = 2$.

We obtain V in the same way from (13) but have to substitute

$$A' = (a_1 + b_1 + 1)^2, \quad B' = B + a_1(a_1 - 1), \quad C' = C$$

for A, B, C.

Putting $a_1 + b_1 + 1 = -m$ so that we have again $A = m^2$ we obtain for the eigenvalues the expression

$$\lambda = (\sigma + a_1 + m + n)(\sigma + a_1 + m + n + 1) + C' \quad \ldots \quad (15')$$

where $\sigma + a_1$ is given by the equation

$$(\sigma + a_1)(\sigma + a_1 - 1) = B. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16')$$

The equations (15′) and (16′) we obtain from (15) and (16) if we write $\sigma + a_1$ instead of σ. But this changes only the notation.

§ 5. *The differential equation of* JACOBI *polynomials.* If SOMMERFELD's polynomial method is not applicable to a certain differential equation, we can try to give the latter a new form by a transformation of the independent variable and then to apply this method. We expect to succeed in this way from the fact, that in SOMMERFELD's polynomial equation (1.2) the zero point plays a distinguished role which is after a transformation taken over by another point of the fundamental interval. That means: If we replace in a "given" differential equation the independent variable by a new one and regard such an obtained equation as the "original" differential equation (2.1) or (2.5) we can generally solve the "given" differential equation by the polynomial method for other potentials V, as by direct application of this method to the "given" differential equation.

To verify this statement we use the differential equation of the associated Legendre functions (4.1) i.e.

$$\frac{d}{dx'}\left((1-x'^2)\frac{dK}{dx'}\right) + (\lambda - V(x'))K = 0 \quad . \quad . \quad . \quad . \quad (1)$$

where we have denoted the independent variable by x'. Substituting here by

$$x' = x - 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

the new independent variable x, we obtain the differential equation

$$\frac{d}{dx}\, x(2-x)\frac{dK}{dx} + (\lambda - V(x))\, K = 0$$

which we will consider as the "original" equation of SOMMERFELD's polynomial method. It has the form (2.5) with

$$p = x(2-x), \quad \varrho = 1, \quad q = V \quad . \quad . \quad . \quad . \quad . \quad (3)$$

so that according to (2.9) and (3) the potential V has the form

$$V = \frac{1}{x(2-x)} + Sx(2-x) + \lambda. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In the variable x' the fundamental interval is bounded by ± 1, in x it is therefore given by $0 < x < 2$.

For further conclusions we use first the factor E. We have to choose it in the form (2.18*a*) to guarantee finiteness of the normalizing integral

$$\int_0^2 K^* K\, dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Otherwise $p = x(2-x)$ endangers the convergence of (5) at its upper limit. Therefore we have to put $A_2 \neq 0$, $B_2 \neq 0$ and obtain

$$E = x^{\frac{A_1}{A_2}-1} \frac{(A_2 + B_2 x^h)^{\frac{1}{h}\left(\frac{B_1}{B_2} - \frac{A_1}{A_2}\right)}}{(2-x)^{1/2}}.$$

To guarantee the convergence of (5) for $x = 2$ we must suppose that the expression $A_2 + B_2 x^h$ is divisible by $2-x$, i.e. that

$$B_2 = -\frac{1}{2^h} A_2. \quad \ldots \ldots \ldots \quad (6)$$

Further we have to assume according to the higher demands of (2. 15) that

$$2\left[\frac{1}{h}\left(\frac{B_1}{B_2} - \frac{A_1}{A_2}\right) - \frac{1}{2}\right] + 1 = \frac{2}{h}\left(\frac{B_1}{B_2} - \frac{A_1}{A_2}\right) > \frac{1}{2}. \quad \ldots \quad (7)$$

Finally the removal of the x-power from E to P gives

$$A_1 = \tfrac{1}{2} A_2. \quad \ldots \ldots \ldots \quad (8)$$

To use V for further considerations we remark that according to (6) and (8) the expression S is given by

$$S = -\frac{(1 + m y^h)(1 + n y^h)}{16\, y^2 (1 - y^h)^2} - \frac{a_0}{4} \frac{1 + p y^h}{y^2 (1 - y^h)}$$

where $y = x/2$ and

$$m = 2^{h+1} b_1, \quad n = -2^{h+1} b_1 - 2(h+1), \quad p = \frac{2^h (b_0 - h b_1)}{a_0}.$$

Developing S in a power series in y we obtain for the expression $Sx(2-x)$ appearing in Eq. (4) for V

$$Sx(2-x) = 4\, Sy(1-y) = -(a_0 + \tfrac{1}{4})\left(\frac{1}{y} - 1\right) + M(y^{h-1} - y^h) +$$
$$+ (M+N)(y^{2h-1} - y^{2h}) + (M + 2N)(y^{3h-1} - y^{3h}) + \ldots$$

where

$$M = \frac{h}{2} - a_0 - 2^h (b_0 - h b_1), \; N = \tfrac{1}{4}\left[2h - 1 + 2^{h+1} b_1 (2^{h+1} b_1 + 2h + 2)\right].$$

To fix the value of h we can now use the demand, that the coefficients of $Sx(2-x) + \lambda$ (cp. (2. 13) and (4)) are independent of λ. For $h = 2, 3, 4 \ldots$ we have to claim that $a_0 + 1/4$, $a_0 + 1/4 + \lambda$, M, N are constant, so that λ would be constant.

For $h = 1$ the expressions

$$a_0 + \tfrac{1}{4}, \quad a_0 + \tfrac{1}{4} + M + \lambda, \quad N$$

only have to be constant. But this means that

$a_0 = \text{const}$, $b_1 = \text{const}$ (9*a*), $\lambda = -M + \text{const} = 2b_0 + \text{const}$ (9*b*)

λ can now depend on quantum numbers because b_0 is not constant now.

To consider the case $h = 1$ in detail we remark that according to (2. 11), (4), (6) and (8) the potential V has the form

$$V = \frac{A}{x-2} + \frac{B}{x} + C$$

where

$A = -8b_1(b_1+1) - 2$ (10a), $B = -2a_0$ (10b), $C = \lambda - 2(b_0 + b_1 + 2b_1^2)$. (10c)

The relations (9a) follow also from (10a) and (10b) and the relation (9b) follows from (10c).

Reintroducing by (2) again the variable x', we obtain V in the form

$$V = \frac{A}{x'-1} + \frac{B}{x'+1} + C = \frac{x'(A+B) + A - B}{x'^2 - 1} + C.$$

But (1) represents with this V the differential equation of JACOBI polynomials. It is therefore situated at the limit of the applicability of SOMMERFELD's polynomial method.

In the quantum theory of a spinning symmetrical top we have to do with this equation with

$$A = -\tfrac{1}{2}(\tau - \tau')^2, \qquad B = \tfrac{1}{2}(\tau + \tau')^2 \quad . \ . \ . \ . \ (11)$$

where τ and τ' are positive or negative integers.

Using that we have according to (2. 19) and (6)

$$(\sigma + n)(\sigma + n - 1) - 4(\sigma + n)b_1 - 2b_0 = 0,$$

we obtain from (10) the eigenvalues

$$\lambda = (\sigma + n - 2b_1)(\sigma + n - 2b_1 - 1) + C.$$

According to (2. 17), (8) and (10) we have to calculate σ from

$$\sigma^2 = -a_0 = -\tfrac{1}{2}B. \quad . \ . \ . \ . \ . \ . \ . \ . \ (12)$$

Supposing especially the case (11) we get from (10) according to (7): $b_1 = -\frac{1}{2} + \frac{|\tau - \tau'|}{2}$ and from (12) in accordance with the fact that $\sigma > 0$ (otherwise we would obtain for $x' = -1$ an inadmissible singularity): $\sigma = \frac{|\tau - \tau'|}{2}$. Hence we get the well known result

$$\lambda = (n + \tau^*)(n + \tau^* + 1) + C$$

where $\tau^* = \frac{|\tau + \tau'|}{2} - \frac{|\tau - \tau'|}{2}$ is the larger of both the integers $|\tau|$ and $|\tau'|$.

REFERENCES.

1. A comprehensive treatment was given by A. SOMMERFELD in Atombau und Spektrallinien, Vol. II, Braunschweig 1939, cp. p. 716.
2. SOMMERFELD denotes our coefficients $2A_1$ and $2B_1$ by A_1 and B_1.
3. (a, b) means equation b of section a.
4. Compare however (3. 1) where q depends on l.