Mathematics. — *A cohomology theory with higher coboundary operators I, (Construction of the Groups *).)* By SZE-TSEN HU. (Communicated by Prof. L. E. J. BROUWER.)

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1. *Introduction* [1]).

The purpose of the present work is to give a generalization of the classical cohomology theory in algebraic topology by introducing coboundary operators of higher order. In the present note, we shall construct such a theory in details by using WALLACE cochains of a topological space, as given in a recent paper of SPANIER, [2] [2]). However, one might easily establish analogous generalisations for the singular homology and cohomology theory and for the homology and cohomology groups of an abstract group.

2. *Cochains of a space.*

Let X be a topological space [3]). Following SPANIER, [2, p. 408], we shall denote by X^{m+1} the $(m+1)$-fold topological product of X with itself and denote by $\triangle X^{m+1}$ the diagonal of X^{m+1}. Let G be a given discrete abelian group, written in the additive form.

By an *m-function* of X into G, we mean a function $\varphi\colon X^{m+1} \to G$ defined on the space X^{m+1} with values in G. Continuity is not assumed. The totality of all m-functions of X into G form a group $\Phi^m(X, G)$ with functional addition as the group operation. An m-function φ of $\Phi^m(X, G)$ is said to be *locally zero* if there is an open set N of X^{m+1} containing $\triangle X^{m+1}$ such that φ is zero on N. It is easy to see that the set of all locally zero m-functions form a subgroup $\Phi_0^m(X, G)$ of $\Phi^m(X, G)$.

The factor group

$$C^m(X, G) = \Phi^m(X, G) / \Phi_0^m(X, G)$$

is defined to be *the group of m-cochains* of X over G, whose elements, the cosets of $\Phi_0^m(X, G)$ in $\Phi^m(X, G)$, will be called the *m-cochains* of X over G. We shall denote by $[\varphi]$ the m-cochain which contains $\varphi \in \Phi^m(X, G)$, i.e.

$$[\varphi] = \varphi + \Phi_0^m(X, G),$$

and φ will be called a *representative* of $[\varphi]$.

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1) The author acknowledges his gratitude to Professor A. D. WALLACE for his kind suggestions.

2) Numbers in brackets refer to the bibliography at the end of the paper.

3) The topological spaces considered here are of the most general type, no separation axioms are postulated.

Further, we define $C^m(X, G) = 0$ if $m < 0$, i.e. the group which consists of a single element.

3. *The p-th coboundary operator.*

Let p be a given positive integer. For any given p integers $i_1, \ldots, i_p$ such that

$$0 \leqq i_1 < i_2 < \ldots < i_p \leqq m + p$$

we define a projection

$$\Pi_{i_1 \ldots i_p} : X^{m+p+1} \to X^{m+1}$$

by taking

$$\Pi_{i_1 \ldots i_p}(x_0, \ldots, x_{m+p}) = (x_{j_0}, \ldots, x_{j_m}),$$

where $(i_1, \ldots, i_p, j_0, \ldots, j_m)$ form a permutation of the $m + p + 1$ integers $0, 1, \ldots, m + p$ with

$$j_0 < j_1 < \ldots < j_m.$$

We shall define the p-th coboundary operator δ^p as follows. For an arbitrary m-function $\varphi \in \Phi^m(X, G)$, its p-th coboundary $\delta^p \varphi$ is an $(m+p)$-function of $\Phi^{m+p}(X, G)$ defined by

$$(\delta^p \varphi)(x_0, \ldots, x_{m+p}) =$$

$$= \Sigma \operatorname{sgn} \begin{pmatrix} 0, \ldots, p-1, & p, \ldots, m+p \\ i_1, \ldots, \; i_p & , j_0, \ldots, \; j_m \end{pmatrix} \varphi(\Pi_{i_1 \ldots i_p}(x_0, \ldots, x_{m+p})).$$

The summation ranges over all the possible choices of the integers $i_1, \ldots, i_p$ such that the integers $(i_1, \ldots, i_p, j_0, \ldots, j_m)$ form a permutation of the $m + p + 1$ integers $0, \ldots, m + p$ satisfying the sectionwise normality condition:

$$i_1 < i_2 < \ldots < i_p, \quad j_0 < j_1 < \ldots < j_m.$$

For a given permutation P of objects, the value of $sgn(P)$ is 1 or -1 according as P is even or odd.

The p-th coboundary operator δ^p, defined above, clearly is a homomorphism

$$\delta^p : \Phi^m(X, G) \to \Phi^{m+p}(X, G).$$

One of its important properties is:

$$\delta^p \textit{ maps } \Phi_0^m(X, G) \textit{ into } \Phi_0^{m+p}(X, G). \quad . \quad . \quad . \quad . \quad (3.1)$$

Proof. Let $\varphi \in \Phi_0^m(X, G)$ be an arbitrary locally zero m-function. There is an open set N of X^{m+1} containing $\triangle X^{m+1}$ on which φ vanishes. Let $N^* \subset X^{m+p+1}$ denote the intersection of all sets $\Pi_{i_1 \ldots i_p}^{-1}(N)$, then $\delta_p \varphi$ is zero on N^*. Since $\Pi_{i_1 \ldots i_p}$ is continuous and maps $\triangle X^{m+p+1}$ into $\triangle X^{m+1}$, each $\Pi_{i_1 \ldots i_p}^{-1}(N)$ is an open set containing $\triangle X^{m+p+1}$. Since there are only a finite number of projections $\Pi_{i_1 \ldots i_p}$ for a given m, N^* is an open set of X^{m+p+1} containing the diagonal $\triangle X^{m+p+1}$. Hence we get $\delta^p \varphi \in \Phi_0^{m+p}(X, G)$, Q.E.D.

It follows from (3. 1) that the p-th coboundary operator δ^p, defined for m-functions, induces a homomorphism

$$\delta^p : C^m(X, G) \to C^{m+p}(X, G) \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

by the relation:

$$\delta^p[\Phi] = [\delta^p \Phi].$$

Further, we define δ^p in (3. 2) to be $\delta^p = 0$ for each $m < 0$.

4. *Successive coboundaries.*

It is natural to ask whether or not the following equality

$$\delta^p \delta^q = 0$$

holds for a given pair of positive integers p and q. To answer this question, we shall calculate the $(m+p+q)$-function $\delta^p \delta^q \Phi$ for an arbitrary m-function $\Phi \in \Phi^m(X, G)$.

By the definitions of δ^q and δ^p in § 3, one can easily deduce that

$$\delta^p \delta^q \Phi(x_0, \ldots, x_{m+p+q}) =$$

$$= \Sigma \operatorname{sgn} \begin{pmatrix} 0, \ldots, p-1, p, \ldots, p+q-1, p+q, \ldots, m+p+q \\ i_1, \ldots, \ i_p \ , j_1, \ldots, \quad j_q \quad , \ k_0 \ , \ldots, \quad k_m \end{pmatrix} \Phi(x_{k_0}, \ldots, x_{k_m})$$

where $i_1, \ldots, i_p, j_1, \ldots, j_q, k_0, \ldots, k_m$ is any permutation of the $m+p+q+1$ integers $0, 1, \ldots, m+p+q$ subjected to the condition that

$$i_1 < i_2 < \ldots < i_p, \quad j_1 < j_2 < \ldots < j_q, \quad k_0 < k_1 < \ldots < k_m,$$

and the summation on the right member ranges over all of these permutations.

Let us consider a fixed choice of the integers $k_0, \ldots, k_m$ such that

$$0 \leqq k_0 < k_1 < \ldots < k_m \leqq m+p+q.$$

Let $h_1, \ldots, h_{p+q}$ be $p+q$ integers such that $h_1 < h_2 < \ldots < h_{p+q}$ and that $h_1, \ldots, h_{p+q}, k_0, \ldots, k_m$ form a permutation of the integers $0, 1, \ldots, m+p+q$. The coefficient of $\Phi(x_{k_0}, \ldots, x_{k_m})$ in our formula for $\delta^p \delta^q \Phi$ is

$$\Sigma \operatorname{sgn} \begin{pmatrix} 0, \ldots, p-1, p, \ldots, p+q-1, p+q, \ldots, m+p+q \\ i_1, \ldots, \ i_p \ , j_1, \ldots, \quad j_q \quad , \ k_0 \ , \ldots, \quad k_m \end{pmatrix} =$$

$$= \left\{ \Sigma \operatorname{sgn} \begin{pmatrix} h_1, \ldots, h_p, h_{p+1}, \ldots, h_{p+q} \\ i_1, \ldots, i_p, \ j_1 \ , \ldots, \ j_q \end{pmatrix} \right\} \operatorname{sgn} \begin{pmatrix} 0, \ldots, p+q-1, p+q, \ldots, m+p+q \\ h_1, \ldots, \ h_{p+q} \ , \ k_0 \ , \ldots, \quad k_m \end{pmatrix}$$

where $i_1, \ldots, i_p, j_1, \ldots, j_q$ is any permutation of the $p+q$ integers $h_1, \ldots, h_{p+q}$ subjected to the condition of sectionwise normality:

$$i_1 < i_2 < \ldots i_p, \quad j_1 < j_2 < \ldots < j_q,$$

and the summations on both sides are taken over all of these permutations.

Then it follows easily that

$$\delta^p \delta^q \Phi = \theta(p, q) \, \delta^{p+q} \Phi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.1)$$

where $\theta(p, q)$ denotes the algebraic sum of the sign's of all (p, q)*-sectionwise normal permutations* of $p+q$ integers $1, 2, \dots, p+q$. A permutation of $1, 2, \dots, p+q$ is said to be (p, q)*-sectionwise normal*, if it is of the form $i_1, \dots, i_p, j_1, \dots, j_q$ such that

$$i_1 < i_2 < \dots < i_p, \quad j_1 < j_2 < \dots < j_q.$$

5. *The evaluation of* $\theta(p, q)$.

To evaluate $\theta(p, q)$, it is easily verified that

$$\theta(p, q) = (-1)^{pq} \theta(q, p) . \qquad (5.1)$$

For an odd p, it follows immediately from (5.1) that $\theta(p, p) = 0$.

As a recurrence formila for calculation, we have

$$\theta(p, q) = \theta(p, q-1) + (-1)^q \theta(p-1, q), \qquad (5.2)$$

whence one can deduce by induction that

$$\theta(p, q) = \frac{([p/2] + [q/2])!}{[p/2]!\,[q/2]!} \qquad (5.3)$$

when pq is even, and $\theta(p, q) = 0$ when pq is odd. In (5.3), the symbol $[x]$ denotes the greatest integer not exceeding x and $0! = 1$. In particular, when p is even,

$$\theta(p, p) = p!/\{(p/2)!\}^2. \qquad (5.4)$$

As a consequence of (5.3), we have

$$\theta(p, q) = \theta(q, p) \geqq 0 \qquad (5.5)$$

for every pair of integers p and q. Hence,

$$\delta^p \delta^q \Phi = \delta^q \delta^p \Phi = \theta(p, q)\, \delta^{p+q} \Phi$$

for every m-function $\Phi \in \Phi^m(X, G)$, and the coboundary operators δ^p, δ^q, δ^{p+q} are related by

$$\delta^p \delta^q = \delta^q \delta^p = \theta(p, q)\, \delta^{p+q}. \qquad (5.6)$$

6. *The* (p, q)*-cohomology groups.*

Throughout the remainder of the paper, let p, q be a given pair of positive integers and let

$$\theta = \theta(p, q) = \theta(q, p).$$

Let G be a discrete abelian group, written in the additive form. Assume that the order of every element of G is a divisor of θ, i.e. $\theta g = 0$ for each $g \in G$.

Consider the two coboundary operators

$$\delta^p : C^m(X, G) \to C^{m+p}(X, G),$$
$$\delta^q : C^{m-q}(X, G) \to C^m(X, G).$$

The cochains $c^m \in C^m(X, G)$ with $\delta^p c^m = 0$ are called *cocycles of order* p; they form a subgroup

$$Z^{m,p}(X, G) \subset C^m(X, G),$$

which is the kernel of δ^p. The cochains $c^m \in C^m(X, G)$ such that $c^m = \delta^q c^{m-q}$ for some $c^{m-q} \in C^{m-q}(X, G)$ are called *coboundaries of order* q; they form a subgroup

$$B^{m,q}(X, G) \subset C^m(X, G),$$

which is the image of δ^q.

It follows from our assumption concerning the order of the elements of G that $\delta^p \delta^q = 0$; hence

$$B^{m,q}(X, G) \subset Z^{m,p}(X, G).$$

The discrete factor group

$$H^m_{(p,q)}(X, G) = Z^{m,p}(X, G)/B^{m,q}(X, G)$$

will be called the m-dimensional (p, q)-*cohomology group* of X over G.

7. *Induced homomorphisms of cochains.*

Let $f: X \to Y$ be a (continuous) map of X into Y. f will also be used to denote the map $f: X^{m+1} \to Y^{m+1}$ defined by

$$f(x_0, \ldots, x_m) = (f(x_0), \ldots, f(x_m)).$$

The following properties are immediate:

$$f(\triangle X^{m+1}) \subset \triangle Y^{m+1}. \quad \ldots \ldots \quad (7.1)$$

$$\Pi_{i_1 \ldots i_p} f = f \Pi_{i_1 \ldots i_p} \quad \ldots \ldots \quad (7.2)$$

For each m-function $\varphi \in \Phi^m(Y, G)$, let us define an m-function $f' \varphi$ of $\Phi^m(X, G)$ by taking

$$f' \varphi(x_0, \ldots, x_m) = \varphi(f(x_0), \ldots, f(x_m)).$$

Then f' is a homomorphism

$$f': \Phi^m(Y, G) \to \Phi^m(X, G).$$

According to a statement of SPANIER, [2, p. 410], f' maps $\Phi^m_0(Y, G)$ into $\Phi^m_0(X, G)$. Hence, f' induces a homomorphism

$$f^{\#}: \; C^m(Y, G) \to C^m(X, G)$$

defined by $f^{\#}[\varphi] = [f' \varphi]$ for every chain $[\varphi] \in C^m(Y, G)$.

The following properties of the homomorphism $f^{\#}$ are immediate:

If $f: X \to X$ is the identity map, then $f^{\#}$ is the identity autumorphism $\ldots \ldots$ (7.3)

If $f: X \to Y$ and $g: Y \to Z$, then $(gf)^{\#} = f^{\#} g^{\#}$. $\ldots$ (7.4)

$$f^{\#} \delta^p = \delta^p f^{\#}. \quad \ldots \ldots \quad (7.5)$$

8. *The relative groups.*

A pair (X, X_0) is defined as a topological space X and a subspace [4])

[4]) By a subspace of a topological space X, we mean any subset of X, not necessarily closed, with the topology obtained by relativisation.

X_0. Let (X, X_0) be a given pair and $\iota: X_0 \to X$ be the identity map. Then the induced homomorphism

$$\iota^{\#} : \quad C^m(X, G) \to C^m(X_0, G)$$

is defined. The kernel of this homomorphism $\iota^{\#}$ is called *the group of m-dimensional cochains of X modulo X_0 over* G, denoted by

$$C^m(X \bmod X_0, G).$$

As a consequence of (7.5), we have:

$$\delta^p C^m(X \bmod X_0, G) \subset C^{m+p}(X \bmod X_0, G) \ . \quad . \quad . \quad (8.1)$$

Let

$$Z^{m,p}(X \bmod X_0, G) = C^m(X \bmod X_0, G) \cap Z^{m,p}(X, G),$$
$$B^{m,q}(X \bmod X_0, G) = \delta^q C^{m-q}(X \bmod X_0, G).$$

Then it follows from (8.1) and our assumption on G that

$$B^{m,q}(X \bmod X_0, G) \subset Z^{m,p}(X \bmod X_0, G).$$

The discrete factor group

$$H^m_{(p,q)}(X \bmod X_0, G) = Z^{m,p}(X \bmod X_0, G)/B^{m,q}(X \bmod X_0, G)$$

is defined to be the m-dimensional *(p, q)-cohomology group of X modulo X_0 over* G. If X_0 is vacuous, then it reduces to $H^m_{(p,q)}(X, G)$.

Appendix A. A modification of the theory.

In our definition of the (p, q)-cohomology groups over G, given in §§ 6 and 8, we assume that $\theta(p, q)g = 0$ for every element g of G. To get rid of this undesirable assumption, a modification of the theory was proposed to the author by Prof. A. D. WALLACE as follows.

Let $\theta(p, q)G$ denote the subgroup of G which consists of all elements of the form $\theta(p, q)g$, $g \in G$. In the definition of cochains given in § 2, we consider instead of $\Phi^m_0(X, G)$, the larger subgroup $\Phi^m_{(p,q)}(X, G)$ of $\Phi^m(X, G)$, which consists of all m-functions ϕ of X into G such that there exists an open set N_ϕ of X^{m+1} containing the diagonal $\triangle X^{m+1}$ with $\phi(x_0, \ldots, x_m) \in \theta(p, q)G$ for every point $(x_0, \ldots, x_m)$ of N_ϕ. Define the factor group

$$C^m_{(p,q)}(X, G) = \Phi^m(X, G)/\Phi^m_{(p,p)}(X, G)$$

to be the group of m-dimensional (p, q)-cochains of X over G. In terms of these cochains, the groups of §§ 6 and 8 can be defined without the assumption that $\theta(p, q)G = 0$.

Appendix B. A sketch of the singular theory.

Let X be a topological space and let $S(X)$ denote the singular complex of X as defined by EILENBERG, [1, p. 420]. Let $C_m(X)$ denote the group of integral singular m-chains in X.

Let p be a given positive integer and consider the ordered geometric m-simplex ($m \geqq p$)

$$s = < v_0, v_1, \ldots, v_m >.$$

From the $m+1$ integers $0, 1, \ldots, m$, let us take arbitrary p distinct integers $i_1, \ldots, i_p$ satisfying

$$i_1 < i_2 < \ldots < i_p.$$

Let the $m-p+1$ remaining integers be denoted by $j_0, \ldots, j_{m-p}$ with the order not disturbed. They determine an $(m-p)$-dimensional face

$$s^{(i_1, \ldots, i_p)} = < v_{j_0}, v_{j_1}, \ldots, v_{j_{m-p}} >$$

of s.

Given a singular m-simplex

$$T\colon\ s \to X,$$

consider the singular $(m-p)$-simplices

$$T^{(i_1, \ldots, i_p)}:\quad s^{(i_1, \ldots, i_p)} \to X$$

defined by the partial mappings $T^{(i_1, \ldots, i_p)} = T \,|\, s^{(i_1, \ldots, i_p)}$. We define the p-th boundary of T to be

$$\partial^p T = \Sigma \operatorname{sgn} \begin{pmatrix} 0, \ldots, p-1, & p, \ldots, & m \\ i_1, \ldots, & i_p & , j_0, \ldots, j_{m-p} \end{pmatrix} T^{(i_1, \ldots, i_p)}$$

The summation ranges over all the possible choices of the integers $i_1, \ldots, i_p$ described above.

It is clear that $T_1 \equiv T_2$ implies that

$$T_1^{(i_1, \ldots, i_p)} \equiv T_2^{(i_1, \ldots, i_p)}$$

and therefore $\partial^p T_1 = \partial^p T_2$ in $C_{m-p}(X)$. Hence we get a homomorphism

$$\partial^p\colon C_m(X) \to C_{m-p}(X).$$

As in § 4, one might verify that

$$\partial^p \partial^q = \partial^q \partial^p = \theta(p, q)\, \partial^{p+q}$$

for any two positive integers p and q. Hence, one may define the *singular (p, q)-homology groups and the singular (p, q)-cohomology groups.*

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