Mathematics. — *The second pearl of the theory of numbers.* By J. G. VAN DER CORPUT and J. H. B. KEMPERMAN. (Second communication.)

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§ 2. *Generalisations of the $A+B$-theorem.*

In theorem 4, treated in the first communication, we have introduced an abstract addition. If we take there for the abstract sum of two numbers the ordinary product of these two numbers, theorem 4 furnishes the AB-theorem; in fact this is obvious if both A and B contain 1, and otherwise we may repeat here the argument by which we have shown in § 1, immediately after the formulation of theorem 4, that the $A+B$-theorem is a special case of theorem 4.

We find another application of theorem 4, if we put the abstract sum of two numbers a and b equal to $a+b+\lambda ab$, where λ is a given positive number. In this manner we obtain:

If γ and δ denote given numbers $\geqq 0$, and g and λ given positive numbers, and if the finite sets A and B formed by numbers $\geqq 0$ satisfy the inequalities

$$A(h)+B(h)\geqq 1+\gamma\log(\lambda h+1)+\delta$$

for $h=g$ and for each positive number $h<g$ belonging to A or B, then

$$(A+B+\lambda AB)(h)\geqq\gamma\log(\lambda h+1)+\delta$$

for each positive number $h\leq g$.

This theorem is an application of theorem 4 except in the case where the number 0 does not belong to both sets A and B; in the latter case the given inequality, applied with h equal to the smallest positive number which either is equal to g or belongs to A or B, gives $\gamma=\delta=0$, since for that value of h we have $A(h)+B(h)\leqq 1$.

We give another application of theorem 4 in which we consider sets formed by complex numbers and where $S[h]$ denotes the number of elements of S with real part $<h$.

Let g be a given positive number and let $\varphi(h)$ be for $h>0$ a slowly increasing and monotonic non-decreasing function.

If the finite sets A and B formed by complex numbers with real part $\geqq 0$ contain both at least one purely imaginary element and satisfy the inequalities

$$A[h]+B[h]\geqq 1+\varphi(h)$$

for each positive number $h\leqq g$, then we have for these h

$$(A+B)[h]\geqq\varphi(h).$$

We may suppose in the proof that both A and B contain the number 0 and that neither A nor B contains an element $-ip$, where $p>0$; in fact if not, it is sufficient to submit A and B to a suitable translation parallel to the imaginary axis. Let us introduce the set G of the complex numbers either with positive real part, either of the form $i p$, where $p \geqq 0$. This set is ordered by the convention $x_1 + i y_1 < x_2 + i y_2$ first if $x_1 < x_2$ and secondly if $x_1 = x_2$, $y_1 < y_2$. We define $\varphi(x + i y)$ on G by the convention

$$\begin{aligned} \varphi(x + i y) &= \varphi(x) \quad \text{for } x > 0, \\ &= 0 \quad\quad\; \text{for } x = 0. \end{aligned}$$

Since the elements of A and B with real part $\geqq g$ do not occur in our proposition, it is not necessary to consider them, so that we may apply theorem 4, where H denotes the set of elements $\leqq g$ of G. This establishes the proof.

Theorem 4 is a generalization of the $A + B$-theorem which is valid for abstract sets G on which a commutative and associative addition has been defined. We have chosen the formulation above of theorem 4, because this generalization of the $A + B$-theorem has the same proof as the $A + B$-theorem itself. Nevertheless we were conscious of the fact that the assumptions of theorem 4 may be replaced by much weaker conditions. We will do this in this communication with the disadvantage that the argument given in the first communication must be replaced by a much more subtle mode of proof. In order to avoid unnecessary complications we restrict us in what follows to the case where $\varphi(h)$ is monotonic non-decreasing on the ordered set G, on which it has been defined.

First we state that it is not necessary to define the sum $a + b$ for each pair of elements a and b belonging to G. It is sufficient to suppose first that, if a is an arbitrary element of A and b an arbitrary element of B, then G contains an uniquely defined element $a + b$. These elements form a subset $A + B$ of G. Moreover we suppose that the sum $g + b$ has been defined uniquely for each element g of $A + B$ and for each element b of B.

The double condition that the addition is commutative and associative may be replaced by the simple condition

$$(a + b) + b' = (a + b') + b$$

valid for each element a of A and each pair of elements b and b' of B.

In view of these facts we say that we have defined on a set G a *partial addition* with respect to the non-empty subsets A and B of G, if for every element b of B the sum $g + b$ denotes an uniquely defined element of G, first for each element g of A and secondly for each element g of $A + B$, such that the following four *addition properties* are valid:

1. If a is an element of A and b and b' are elements of B, then
$$(a + b) + b' = (a + b') + b.$$
2. If b is an element of B, and g and g' belong either both to A or both to $A + B$, then
$$g + b = g' + b \text{ implies } g = g'.$$

3. If a is an element of A, and b and b' are elements of B, then
$$a + b = a + b' \text{ implies } b = b'.$$
4. B contains an element b_0 such that it is impossible to find in B an element $b \neq b_0$ and in $A + B$ a finite number of elements
$$g_0, g_1, \ldots, g_{m-1}, g_m = g_0 \qquad (\text{where } m \geqq 1)$$
with the properties
$$g_\mu + b = g_{\mu+1} + b_0 \qquad (\mu = 0, 1, \ldots, m-1).$$

Using this convention we may generalise theorem 4, apart from the monotony of $\varphi(h)$, in the following manner:

Theorem 5. Condition 1. *Let be defined on an ordered set G a partial addition with respect to two given finite non-empty subsets A and B of G. We denote the smallest element of A and B respectively by a_0 and b_0, where b_0 is the particular element of B mentioned in the fourth addition property.*

Let be given on G a real monotonic non-decreasing function $\varphi(g)$ with
$$\varphi(g) = \varphi(g + b_0); \ \varphi(b) \leqq \varphi(g + b); \ \varphi(b) = \varphi(a_0 + b),$$
where g is an arbitrary element of A or $A + B$ and where b is an arbitrary element $\neq b_0$ of B.

Further we assume $\varphi(a_0) \leqq 0$ if $A + B'$ contains the element $a_0 + b_0$ and also if $A + B'$ contains at least one element $< a_0$; here B' is the set formed by the elements $\neq b_0$ of B.

Condition 2. *Finally we assume*
$$\varphi(g + b) \leqq \varphi(g) + \varphi(b),$$
where g is an arbitrary element of A or $A + B$ and where b is an arbitrary element $\neq b_0$ of B.

Assertion. *If the inequalities*
$$A(h) + B'(h) \geqq \varphi(h)$$
are valid for each element h belonging to a given subset H of G, which contains every element $> a_0$ of A and every element $> b_0$ of B, then we have for each element $h > a_0 + b_0$ of H
$$(A + B)(h) \geqq \varphi(h).$$

Remark. The inequality to be proved is obvious for the elements $h \leqq a_0$ of H, since they satisfy $\varphi(h) \leqq 0$, as we will show now.

If B' is empty, and also if $h \leqq b'$, where b' denotes the smallest element of B', we have $B'(h) = 0$, and moreover $A(h) = 0$ by $h \leqq a_0$, hence
$$0 = A(h) + B'(h) \geqq \varphi(h)$$
Let us now consider the elements h of H with $b' < h \leqq a_0$. In virtue of the monotony it is sufficient to prove $\varphi(a_0) \leqq 0$. This inequality follows from

condition 1 of theorem 5, if $A + B'$ contains an element $< a_0$, so that we may suppose $a_0 + b' \geqq a_0$, and in that case we obtain

$$\varphi(a_0) \leqq \varphi(a_0 + b') = \varphi(b') \leqq A(b') + B'(b') = 0.$$

If a_0 is an element of H, the inequality of theorem 5 to be proved is valid for each element h of H, since in that case we just have found $\varphi(a_0) \leqq 0$ and therefore $\varphi(a_0 + b_0) = \varphi(a_0) \leqq 0$, hence $\varphi(h) \leqq 0$ for each element $h \leqq a_0 + b_0$ of H.

Theorem 4 with the supplementary condition that $\varphi(g)$ is monotonic non-decreasing is a particular case of the proposition 5, since in that case the conditions of the latter theorem are satisfied. In fact, it is obvious that the properties 1, 2 and 3 mentioned in the definition of the partial addition are valid, and from the fact that $b > 0$ implies $g + b > g$, it follows that also property 4 is valid.

Till now we have restricted us to theorems without weights, but we may give to each element g of G a weight $f(g) \geqq 0$. In that case $S(h)$ does not denote the number of elements $s < h$ of S, but the sum of the weights $f(s)$ of these elements s; in formula

$$S(h) = \sum_{s<h} f(s). \qquad (14)$$

By putting $f(g) = 1$ the following theorem transforms into theorem 5.

Theorem 6. Condition 1. *Let us suppose that condition 1 of theorem 5 is fulfilled.*

Condition 2. *We give to each element g of G a weight $f(g) \geqq 0$, such that*

$$f(g) = f(g + b_0); \quad f(b) \leqq f(g + b),$$

where b is an arbitrary element $\neq b_0$ of B and where g is an arbitrary element of A or $A + B$.

Condition 3. *Further we assume*

$$\varphi(g + b) \leqq \varphi(g) + \varphi(b); \quad f(g + b) \geqq f(g)$$

where b is an arbitrary element $\neq b_0$ of B and where g is an arbitrary element of A or $A + B$.

The assertion is the same as in theorem 5, apart from the fact, that $A(h)$, $B'(h)$ and $(A + B)(h)$ are defined by (14).

In order to generalise this result it is recommendable to introduce not only one pair of functions $\varphi(g)$, $f(g)$, but a finite or infinite number of such pairs. In this manner we obtain the following theorem, which we consider as the principal proposition of these communications, the proof of which will be given in § 3.

Theorem 7. *Suppose that on a given ordered set G with two given non-empty finite subsets A and B a finite or infinite number of pairs of functions $\varphi(g)$ and $f(g)$ have been defined, such that the conditions 1 and 2 of theorem 6 are satisfied for each of these considered pairs.*

If $\varphi(g)$, $f(g)$ is an arbitrary considered pair of functions and if b denotes an arbitrary element $\neq b_0$ of B, then we suppose moreover that it is possible to find among the considered pairs of functions at least one pair, $\Phi(g)$, $F(g)$ (depending on b and on the choice of the pair $\varphi(g)$, $f(g)$) such that

$$\varphi(g+b) \leqq \Phi(g) + \varphi(b); \quad f(g+b) \geqq F(g)$$

for each element g of A or $A+B$.

If under these conditions each pair of functions $\varphi(g)$, $f(g)$ satisfies the inequality

$$A(h) + B(h) \geqq \varphi(h)$$

for each element h of a given subset H of G which contains all elements $> a_0$ of A and all elements $> b_0$ of B, then we have for each considered pair of functions $\varphi(g)$, $f(g)$ and for each element $h > a_0 + b_0$ of H

$$(A+B)(h) \geqq \varphi(h).$$

Of course $A(h)$ etc. is defined by (14). The remark following immediately after theorem 5 is here also true.

As an application we give:

Theorem 8. *Let G be an ordered set containing a smallest element denoted by 0, on which a commutative and associative addition has been defined, with $g+0=g$ and $g+g^* > g$ for $g^* > 0$, such that*

$$g + g' = g + g'' \text{ implies } g' = g''.$$

Let $\varphi^*(g)$ and $f^*(g) \geqq 0$ be monotonic non-decreasing functions on G and let H be a subset of G.

If the finite subsets A and B of H, containing both the element 0, satisfy for each element g of G and each positive element h of H the inequality

$$\sum_{a<h} f^*(a+g) + \sum_{0<b<h} f^*(b+g) \geqq \varphi^*(h+g) - \varphi^*(g), \quad . \; . \; (15)$$

then we have for these element h and g also

$$\sum_{a+b<h} f^*(a+b+g) \geqq \varphi^*(h+g) - \varphi^*(g).$$

P r o o f. For each pair of elements g and g^* of G we put

$$\varphi(g) = \varphi^*(g+g^*) - \varphi^*(g^*) \text{ and } f(g) = f^*(g+g^*),$$

so that $\varphi(g)$ and $f(g)$ depend not only on g, but also on g^*. In order to prove that the conditions of theorem 7 are satisfied for these pairs $\varphi(g)$, $f(g)$, it is sufficient to observe that the pair formed by the functions

$$\Phi(g) = \varphi^*(g + (g^*+b)) - \varphi^*(g^*+b) \text{ and } F(g) = f^*(g + (g^*+b))$$

possess the required properties

$$\varphi(g+b) \leqq \Phi(g) + \varphi(b) \text{ and } f(g+b) \geqq F(g);$$

these relations are valid even with the sign of equality. Now the assertion of theorem 7 furnishes theorem 8.

Let us show that theorem 4 with the supplementary condition that $\varphi(g)$ is a monotonic non-decreasing function is a particular case of theorem 8. We remark that if the conditions of theorem 4 with the monotonic non-decreasing function $\varphi(g)$ are satisfied, they remain valid if $\varphi(g)$ is replaced by

$$\varphi^*(g) = \max.\ (0, \varphi(g)).$$

In fact it is sufficient to prove that the monotonic function $\varphi^*(g)$ is slowly increasing on the set of positive elements of G, that is

$$\varphi^*(g_1 + g_2) \leqq \varphi^*(g_1) + \varphi^*(g_2)$$

for $g_1 > 0$ and $g_2 > 0$, and this inequality is obvious as we have either

$$\varphi^*(g_1 + g_2) = 0 \leqq \varphi^*(g_1) + \varphi^*(g_2)$$

or

$$\varphi^*(g_1 + g_2) = \varphi(g_1 + g_2) \leqq \varphi(g_1) + \varphi(g_2) \leqq \varphi^*(g_1) + \varphi^*(g_2).$$

From this result it follows, that the conditions of theorem 8 are satisfied with $f^*(g) = 1$, and with $\varphi^*(0) = 0$; further

$$A(h) + B'(h) \geqq \varphi^*(h) \geqq \varphi^*(h + g) - \varphi^*(g)$$

for every positive element h of H and for every element g of G. Therefore (15) is satisfied, so that the assertion of theorem 8, applied with $g = 0$, gives

$$(A + B)(h) \geqq \varphi^*(h) \geqq \varphi(h)$$

for every positive element h of H.

In theorem 8 we assume that the inequality (15) is valid for each element g of G and each positive element h of H. That in some cases it is not necessary to assume all these inequalities, appears from the following example. The special case, where $\gamma(h)$ is constant and where H is the set of non-negative integers below a given bound, has been proved by J. G. VAN DER CORPUT [8]).

Let G be a set of numbers $\geqq 0$, such that on G the addition is always possible, and let $f(g)$ be a positive monotonic non-decreasing function on G. Let $\gamma(h)$ be a monotonic non-increasing function $\geqq 0$ defined on a given finite subset H of G.

Further we suppose

$$f(h+g)\,f(h') \geqq f(h'+g)\,f(h) \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

[8]) On sets of integers, Proc. Kon. Ned. Akad. v. Wetensch., Amsterdam, **50**, 345 (1947). Also: Indagationes Mathematica **9**, 203 (1947). VAN DER CORPUT assumes $f^2(h+1) \geqq f(h)\,f(h+2)$ $(h = 1, 2, \ldots)$ in stead of (16). It is clear that his assumption implies (16) for $g = 1, 2, \ldots$; in fact $\frac{f(h+1)}{f(h)}$ and therefore also

$$\frac{f(h+g)}{f(h)} = \frac{f(h+g)}{f(h+g-1)} \cdots \frac{f(h+1)}{f(h)} \qquad (g = 1, 2, \ldots)$$

are monotonic non-increasing functions of h.

for every element g of G and every pair of elements h and h' of H, such that $h < h'$.

If both subsets A and B of H contain the number 0 and satisfy the inequality

$$\sum_{a<h} f(a) + \sum_{0<b<h} f(b) \geqq \sum_{h'<h} \gamma(h') f(h') \quad . \quad . \quad . \quad . \quad (17)$$

for each positive element h of H, then we have for these h

$$\sum_{a+b<h} f(a+b) \geqq \sum_{h'<h} \gamma(h') f(h'); \quad . \quad . \quad . \quad . \quad . \quad (18)$$

the sum $\sum_{h'<h}$ is extended over all elements $h' < h$ of H.

Proof. It is sufficient to prove that the inequality (17), valid for each positive element h of H, implies

$$\sum_{a<h_1} f(a+g) + \sum_{0<b<h_1} f(b+g) \geqq \sum_{h'<h_1} \gamma(h'+g) f(h'+g) \quad . \quad . \quad (19)$$

for each positive element h_1 of H and each element g of G. In fact the right hand side of (19) is equal to $\varphi^*(h_1+g) - \varphi^*(g)$, where $\varphi^*(g)$ denotes the function $\sum_{h'<g} \gamma(h') f(h')$, which is monotonic non-decreasing on G; the conditions of theorem 8 are therefore fulfilled, so that the assertion of that theorem, applied with $g = 0$, furnishes the inequality (18) for every positive element h of H.

We put for each positive element $h \leqq h_1$ of H

$$\begin{aligned} \varrho_h &= \frac{f(h^*+g)}{f(h^*)} - \frac{f(h+g)}{f(h)} && \text{for } h < h_1 \\ &= \frac{f(h^*+g)}{f(h^*)} && \text{for } h = h_1, \end{aligned}$$

where h^* denotes the greatest element $< h$ of the finite set H (such an element exists, since h is greater than the element 0 of H). As the positive function $f(h)$ satisfies the inequality (16), we have $\varrho_h \geqq 0$.

From

$$\sum_{a<h\leqq h_1} \varrho_h = \frac{f(a+g)}{f(a)}$$

valid for each element $a \geqq 0$ and $< h_1$ of A, we deduce

$$\sum_{h\leqq h_1} \varrho_h \sum_{a<h} f(a) = \sum_{a<h_1} f(a) \sum_{a<h\leqq h_1} \varrho_h = \sum_{a<h_1} f(a+g),$$

and similarly

$$\sum_{h\leqq h_1} \varrho_h \sum_{0<b<h} f(b) = \sum_{0<b<h_1} f(b+g).$$

Moreover

$$\sum_{h\leqq h_1} \varrho_h \sum_{h'<h} \gamma(h') f(h') = \sum_{h'<h_1} \gamma(h') f(h') \sum_{h'<h\leqq h_1} \varrho_h =$$

$$= \sum_{h'<h_1} \gamma(h') f(h'+g) \geqq \sum_{h'<h_1} \gamma(h'+g) f(h'+g).$$

Multiplying both members of (17) with $\varrho_h \geqq 0$ and adding the thus found inequalities, we obtain the inequality (19), which was to be proved.

Although the set G occurring in theorem 6 is supposed to be ordered, it is nevertheless possible to apply that proposition on sets for which no order has been defined, for instance:

Theorem 9. *Consider a commutative additive semigroup* $\mathfrak{G}$ *without any finite non-trivial sub-semigroup. Define* $f(g) \geqq 0$ *and* $\varphi(g)$ *on* $\mathfrak{G}$ *with* $\varphi(0) = 0$ *and*

$$\varphi(g) \leqq \varphi(g+g') \leqq \varphi(g) + \varphi(g'); \quad f(g) \leqq f(g+g'),$$

where g *and* g' *denote arbitrary elements of* $\mathfrak{G}$.

Let σ *be positive. If the finite subsets* A *and* B *of* $\mathfrak{G}$ *contain both the element* 0 *and satisfy the inequalities*

$$\sum_{\varphi(a)<\varrho} f(a) + \sum_{\substack{\varphi(b)<\varrho \\ b \neq 0}} f(b) \geqq \varrho \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

for each number $\varrho \leqq \sigma$, *then we have for these numbers* ϱ

$$\sum_{\varphi(a+b)<\varrho} f(a+b) \geqq \varrho.$$

Remark. We call a set $\mathfrak{H}$, on which an associative addition has been defined, a semigroup, if it contains an element 0 with $h + 0 = 0 + h = h$, valid for each element h of $\mathfrak{H}$, and if further both $h + h' = h + h''$ and $h' + h = h'' + h$ imply $h' = h''$. The trivial sub-semigroup is formed by the element 0.

Proof. Without loss of generality we may suppose that $\varphi(g)$ is everywhere $\geqq 0$ and $\leqq \sigma$, for otherwise we introduce $\varphi^*(g)$ with

$$\begin{aligned} \varphi^*(g) &= 0 && \text{if} \quad \varphi(g) \leqq 0 \\ \varphi^*(g) &= \varphi(g) && \text{if} \quad 0 \leqq \varphi(g) \leqq \sigma \\ \varphi^*(g) &= \sigma && \text{if} \quad \varphi(g) \geqq \sigma, \end{aligned}$$

in stead of $\varphi(g)$, just as we have done in the proof that theorem 4 is a consequence of theorem 8.

We apply theorem 6. If $\mathfrak{G}$ contains at least one element g^* with $\varphi(g^*) = \sigma$, then we choose in this theorem $G = \mathfrak{G}$. Otherwise we denote by G the set, formed by the elements of the given semigroup $\mathfrak{G}$ and further a new element g^*; in this case we put $\varphi(g^*) = \sigma$ and we choose $f(g^*)$ arbitrarily $\geqq 0$.

On G we define a transitive order, such that 0 is the smallest element of G and that $\varphi(g)$ is monotonic non-decreasing on G.

First we show that we have on G a partial addition with respect to the given subsets A and B. For that purpose it is sufficient to examine the fourth addition property. Suppose therefore that the relations ($m \geqq 1$)

$$g_\mu + b = g_{\mu+1} \qquad (\mu = 0, 1, \ldots, m-1)$$

with $g_0 = g_m$ are valid, where $b \neq 0$ and $g_0, \dots g_m$ denote elements of the commutative semigroup $\mathfrak{G}$. Then we have $m\,b = 0$, and this is impossible, since $\mathfrak{G}$ does not possess a finite non-trivial sub-semigroup.

The conditions of theorem 6 are satisfied with $H = G$. In order to apply the assertion of that theorem, we must deduce the inequality

$$\sum_{a<h} f(a) + \sum_{0<b<h} f(b) \geqq \varphi(h) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

for each element h of G, and this is obvious as $\varphi(a) < \varphi(h)$ implies $a < h$ and (21) follows from (20), applied with $\varrho = \varphi(h) \leqq \sigma$.

The assertion of theorem 6 gives

$$\sum_{a+b<h} f(a+b) \geqq \varphi(h)$$

for each element h of G. To show that this result implies the inequality which is to be proved, we introduce the smallest element h^* with $\varphi(h^*) \geqq \varrho$, which is equal to g^* or belongs to $A + B$; such an element exists by $\varphi(g^*) = \sigma \geqq \varrho$. From this definition of h^* it follows that each element $a + b < h^*$ of $A + B$ satisfies the inequality $\varphi(a + b) < \varrho$, hence

$$\sum_{\varphi(a+b)<\varrho} f(a+b) \geqq \sum_{a+b<h^*} f(a+b) \geqq \varphi(h^*) \geqq \varrho.$$