Mathematics. — *On the small vibrations of non-holonomic systems.* By O. BOTTEMA. (Communicated by Prof. W. VAN DER WOUDE.)

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The small vibratory motions of a conservative non-holonomic system about a position of equilibrium have been considered by WHITTAKER in his *Analytical Dynamics* [1]. His conclusions are that here "the difference between holonomic and non-holonomic systems is unimportant" and that the vibratory motion of a given non-holonomic system with n independant coordinates and m non-integrable kinematical relations is the same as that of a certain holonomic system with $n—m$ degrees of freedom.

These conclusions appear to be incorrect.

For the sake of simplicity we first consider a system with *three* coordinates x_1, x_2, x_3, kinetic energy

$$T = \tfrac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and potential energy $V(x_1, x_2, x_3)$. Suppose that there is *one* kinematical relation, integrable or non-integrable:

$$A_1 \dot{x}_1 + A_2 \dot{x}_2 + A_3 \dot{x}_3 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where A_i $(i = 1, 2, 3)$ are functions of x_1, x_2 and x_3.

Suppose further that $x_1 = x_2 = x_3 = 0$ is a position of equilibrium. This means: if for $t = 0$ we have $x_1 = x_2 = x_3 = \dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$ then the system is permanently at rest. What can be said then about $V(x_1 x_2 x_3)$? We write down the Lagrangian equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} - \frac{\partial T}{\partial x_i} = -\frac{\partial V}{\partial x_i} + \lambda A_i$$

or

$$\ddot{x}_i = -\frac{\partial V}{\partial x_i} + \lambda A_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where λ is a function of t only. Substitution of $x_i = 0$ gives

$$\frac{\partial V(0, 0, 0)}{\partial x_i} = \lambda_0 A_i(0, 0, 0) . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence for $x_1 = x_2 = x_3 = 0$ the $\partial V / \partial x_i$ are proportional to the coefficients A_i (which is of course obvious in view of the dynamical meaning of

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[1]) E. T. WHITTAKER, A treatise on the Analytical Dynamics of particles and rigid bodies, fourth revised edition (1936); first american printing, New York (1944), pg. 221—226.

$\partial V/\partial x_i$ and the geometrical meaning of (2)). So for $x_i = 0$ it is not necessary that $\partial V/\partial x_i = 0$, as WHITTAKER supposes.

We may take $\lambda_0 = 1$ without loss of generality in view of the fact that the left member of the kinematical relation (2) can be multiplied with the constant factor λ_0.

Now we consider the motion of the system in the neighbourhood of $x_i = 0$, so that the x_i are small.

We have then, in view of (4):

$$V = a_1 x_1 + a_2 x_2 + a_3 x_3 + \tfrac{1}{2} \Sigma b_{ij} x_i x_j \qquad (b_{ij} = b_{ji}) \quad . \quad . \quad (5)$$

and the kinematical relation (2) is

$$\left.\begin{array}{r}(a_1 + c_{11} x_1 + c_{12} x_2 + c_{13} x_3)\dot{x}_1 + \\ + (a_2 + c_{21} x_1 + c_{22} x_2 + c_{23} x_3)\dot{x}_2 + (a_3 + c_{31} x_1 + c_{32} x_2 + c_{33} x_3)\dot{x}_3 = 0\end{array}\right\} \quad (6)$$

so that the equations of motion are as follows

$$\left.\begin{array}{l}\ddot{x}_1 + a_1 + b_{11} x_1 + b_{12} x_2 + b_{13} x_3 - \lambda(a_1 + c_{11} x_1 + c_{12} x_2 + c_{13} x_3) = 0 \\ \ddot{x}_2 + a_2 + b_{21} x_1 + b_{22} x_2 + b_{23} x_3 - \lambda(a_2 + c_{21} x_1 + c_{22} x_2 + c_{23} x_3) = 0 \\ \ddot{x}_3 + a_3 + b_{31} x_1 + b_{32} x_2 + b_{33} x_3 - \lambda(a_3 + c_{31} x_1 + c_{32} x_2 + c_{33} x_3) = 0\end{array}\right\} . \quad (7)$$

If we put $\lambda = 1 - \varepsilon$, then ε is small and of the same order as x_i. Taking into account in (6) and (7) linear terms only, we have

$$a_1 \dot{x}_1 + a_2 \dot{x}_2 + a_3 \dot{x}_3 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

and

$$\ddot{x}_1 + b_{11} x_1 + b_{12} x_2 + b_{13} x_3 + \varepsilon a_1 - (c_{11} x_1 + c_{12} x_2 + c_{13} x_3) = 0 \quad (7a)$$

(6*a*) and (7*a*) form a system of four lineair homogeneous differential equations for the four unknown functions x_1, x_2, x_3, ε. Substituting $x_i = C_i e^{pt}$, $\varepsilon = C_0 e^{pt}$ and eliminating the constants C we find an equation for p. One of the roots is $p = 0$ and the others are the roots of

$$D \equiv \begin{vmatrix} b_{11} - c_{11} + p^2 & b_{12} - c_{12} & b_{13} - c_{13} & a_1 \\ b_{21} - c_{21} & b_{22} - c_{22} + p^2 & b_{23} - c_{23} & a_2 \\ b_{31} - c_{31} & b_{32} - c_{32} & b_{33} - c_{33} + p^2 & a_3 \\ a_1 & a_2 & a_3 & 0 \end{vmatrix} = 0, \quad . \quad (8)$$

which is of the second degree in p^2, so that we have in all five roots p. The root $p = 0$ is of course an unpleasant one and asks for a further investigation of the system if one inquires about the stability of equilibrium.

As for the other roots, the determinant (8) is in general *not symmetrical* because c_{ij} is not necessarily equal to c_{ji} and here we have a difference with the holonomic case. In order to prove that the difference is essential we give the following example. Let $a_1 = a_2 = 0$, $a_3 = a \neq 0$, $C_{ij} = 0$ with the exception of $c_{21} = c \neq 0$, so that we have the system

$$V = a x_3 + \tfrac{1}{2} \Sigma b_{ij} x_i x_j$$

and the relation

$$c x_1 \dot{x}_2 + a \dot{x}_3 = 0$$

which is non-holonomic. The equation (8) is now

$$p^4 + (b_{11} + b_{22})\,p^2 + b_{11}\,b_{22} - b_{12}\,(b_{12} - c) = 0$$

and its discriminant

$$d = (b_{11} - b_{22})^2 + 4\,b_{12}^2 - 4\,cb_{12}$$

is obviously not always positive. We know however that for a holonomic system, stable or unstable, the values of p^2 are always real.

If the system is holonomic the relation (6) is integrable, hence $c_{ij} = c_{ij}$ and (8) is symmetrical.

We have to consider the case that (6) is not integrable as it stands but can be made so by as integrating factor. If we multiply the left member by $(1 + q_1x_1 + q_2x_2 + q_3x_3 + \ldots)$ and take into account only linear terms, we get

$$c'_{ij} = c_{ij} + a_i\,q_j$$

and it is easily seen that D' is obtained from D by multiplying the last column by q_1 and adding it to the j-th column. Hence we have: for holonomic systems D is symmetrical or can be made so by elementary transformations; for non-holonomic systems D is essentially unsymmetrical.

For the general case of n coordinates x_i and m kinematical relations, where $x_i = 0$ is a position of equilibrium, we have

$$T = \tfrac{1}{2}\,\Sigma\,a_{ij}\,\dot{x}_i\,\dot{x}_j,\quad V = \Sigma\,a_i\,x_i + \Sigma\,b_{ij}\,x_i\,x_j \qquad (a_{ij} = a_{ji},\, b_{ij} = b_{ji})$$

$$\sum_{ij}(a_{ik} + c_{ijk}\,x_j)\,\dot{x}_i = 0 \qquad (k = 1, 2, \ldots m).$$

The equation for the frequencies has m roots $p = 0$ and the other $2(n—m)$ roots are those of

$$D \equiv \begin{vmatrix} b_{11} - \Sigma c_{11k} + a_{11}p^2 & b_{12} - \Sigma c_{12k} + a_{12}p^2 \ldots b_{1n} - \Sigma c_{1nk} + a_{1n}p^2 & a_{11}\; a_{12} \ldots a_{1m} \\ b_{21} - \Sigma c_{21k} + a_{21}p^2 & b_{22} - \Sigma c_{22k} + a_{22}p^2 \ldots b_{2n} - \Sigma c_{2nk} + a_{2n}p^2 & a_{21}\; a_{22} \ldots a_{2m} \\ \cdots & \cdots & \cdots \\ b_{n1} - \Sigma c_{n1k} + a_{n1}p^2 & b_{n2} - \Sigma c_{n2k} + a_{n2}p^2 \ldots b_{nn} - \Sigma c_{nnk} + a_{nn}p^2 & a_{n1}\; a_{n2} \ldots a_{nm} \\ a_{11} & a_{21} \qquad\qquad a_{n1} & 0 \;\; 0 \ldots 0 \\ a_{12} & a_{22} \qquad\qquad a_{n2} & 0 \;\; 0 \ldots 0 \\ \cdots & \cdots & \cdots \\ a_{1m} & a_{2m} \qquad\qquad a_{nm} & 0 \;\; 0 \ldots 0 \end{vmatrix} = 0.$$

For holonomic systems D is symmetrical or can be made so by elementary transformations. This form of the determinant, consisting of an ordinary symmetrical p^2-determinant of the n-th order, to which are added m rows and m columns of constants is given by ROUTH [2]), who, however, takes the relations in integrated form so that the m roots $p = 0$ do not occur.

For non-holonomic systems D is essentially non-symmetrical.

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[2]) ROUTH, The advanced part of a treatise on the dynamics of a system of rigid bodies, Chapter II.