Mathematics. — *A new characterisation of spheres of even dimension* [1]).
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(Communicated at the meeting of May 28, 1949.)

It is the aim of this paper to give a new characterisation of spheres of even dimension. We shall show that a simply-connected manifold is an even sphere if and only if its EULER characteristic is equal to two and it admits transitively a compact transformation group R. In the course of the proof, all the possible groups R are eventually determined. We find that R is either the orthogonal group or the CARTAN's exceptional group G_2, and that G_2 presents only when the manifold is six-dimensional. In an interesting paper [5], MONTGOMERY and SAMELSON have shown that the only compact transitive transformation group of a sphere S^{2n} of $2n$ dimension is the orthogonal group when $2n \geq 114$. By an entirely different method, we fill the gap they left. Furthermore, as an incident result of our discussions, we obtain the first four homotopy groups of the exceptional group G_2.

1. The chief weapon used in this paper is a finite group associated with a connected compact LIE group. This finite group has been fully discussed by various authors. In this section, we shall give a brief sketch of STIEFEL's results [7, §§ 2, 3] which will be used later.

Let R be a connected compact LIE group of dimension r and rank l. All the maximal toral subgroups of R have the same dimension l. Choose one of them, say T. Each normaliser a of T induces an automorphism φ_a: $t \to ata^{-1}$ $(t \varepsilon T)$ of T. All such automorphisms form a finite group which. up to an isomorphism, depends only on R and not on the particular choice of T. We shall denote it by $\Phi(R)$.

Let $U(e)$ be a small neighbourhood of the identity e of R such that it is covered by the canonical coordinates $\xi_1, \xi_2, \ldots, \xi_r$ of the first kind. These coordinates define an r-dimensional tangent euclidean space E^r of R. Each inner automorphism

$$x \to bxb^{-1} \qquad (x \varepsilon R)$$

of R indues a linear transformation S_b of the tangent space E^r which we call the adjoint linear transformation. Now let us consider the maximal toral subgroup T. Its tangent space E^l is a linear subspace of E^r and is l-dimensional. Evidently, the adjoint linear transformation S_a of each normaliser a of T leaves E^l invariant, and hence induces a linear trans-

[1]) The present paper is the revised form of the second part of the author's Ph. D. thesis accepted by Manchester University, 1948.

formation φ'_a of E^l. All these φ'_a form a finite linear group $\Phi'(R)$ isomorphic with $\Phi(R)$.

In the linear space E^l, there are m pencils of parallel hyperplanes, called *singular hyperplanes*, where

$$r = 2m + l. \qquad (1.1)$$

Through the origin there pass *exactly* m singular hyperplanes [3, p. 67]. Each transformation φ' of $\Phi'(R)$ carries singular hyperplanes to singular hyperplanes. Furthermore, we have [7, p. 363].

(1.2) *Let P be a point in E^l. If P is left unaltered by a transformation $\varphi' \varepsilon \Phi'(R)$ other than the identity transformation, then there exists a singular hyperplane passing through both P and the origin.*

It is well-known that compact simple LIE groups fall into four main classes A_k, B_k, C_k, D_k ($k = 1, 2, \ldots$) and five exceptional cases G_2, F_4, E_6, E_7, E_8 where the lower index denotes the rank. In fact, each of the above represents a class of locally isomorphic connected groups among which one is simply-connected and one without centre. In what follows, we shall occasionally use the term "an A_k ($B_k, C_k, \ldots, E_8$)" which simply means any group of the class A_k ($B_k, C_k, \ldots, E_8$). The finite groups Φ associated with these simple groups have been completely determined.

For any connected compact LIE group R, let us denote by $o(R)$ the order of the finite group $\Phi(R)$. Then [9]

$$\left.\begin{array}{l} o(\text{toral group}) = 1,\ o(A_k) = (k+1)!,\ o(D_k) = k! \cdot 2^{k-1}, \\ o(B_k) = o(C_k) = k! \cdot 2^k,\ o(G_2) = 12,\ o(F_4) = 3^2 \cdot 2^7, \\ o(E_6) = 6! \cdot 3 \cdot 2^6,\ o(E_7) = 9! \cdot 2^3,\ o(E_8) = 10! \cdot 3 \cdot 2^6. \end{array}\right\} \qquad (1.3)$$

2. A topological space W is called *homogeneous* if it admits transitively a topological group R of transformations. Let q be a point of W. All the transformations of R which leave q invariant form a closed subgroup L of R called the *isotropic subgroup*. The space W can be regarded as the space R/L of left cosets. R is said to be *effective* (*almost effective*) on W if only the identity (only a finite number of elements) of R preserves every point of W. Suppose that R is not effective on W. Then the elements of R which leave unaltered each point of W form an invariant subgroup I of R. The factor group R/I then acts effectively and transitively on W. Thus without loss of generality of the homogeneous space, we can assume that it admits an effective transformation group.

(2.1) *Let W be a homogeneous manifold of a connected compact group. If W has non-vanishing* EULER *characteristic, then it admits, transitively and almost effectively, a connected, simply-connected compact semi-simple* LIE *group R.*

Proof. By hypothesis, W admits, effectively and transitively, a connected compact group R'. Since W is locally euclidean and R' is compact, it follows that R' must be a LIE group [10, § 7]. Moreover, the EULER characteristic of W does not vanish so that R' is simi-simple and has no centre [8, (2.1), (2.2)] [2]. Therefore the universal covering group R of R' is simply-connected, compact and semi-simple and has only a finite number of centres [6, p. 271]. This group R acts transitively on W in the natural manner. It is easy to see that an element b of R leaves unaltered every point of W if and only if b is a centre of R. Hence R acts almost effectively on W. This group R possesses all the required properties. Proposition (2.1) is therefore proved.

Theorem I. *Let W be a simply-connected manifold with* EULER *characteristic equal to one. If W admits transitively a connected compact transformation group, then W is a single point.*

Proof. By (2.1) there exists a connected, simply-connected compact semi-simple LIE group R acting on W transitively and almost effectively. Therefore, W can be regarded as a coset space R/L where L is a closed subgroup of R. From the simply-connectedness of both R and W, it follows that L is connected [1, § 31].

Since the EULER characteristic $\chi(W) = \chi(R/L) = 1 \neq 0$, R and L have the same rank l [4]. Hence a maximal toral subgroup T of L is, at the same time, a maximal toral subgroup of R. Let E^l be the tangent space of T. We have two finite linear groups $\Phi'(L)$, $\Phi'(R)$ of transformations of the space E^l. From the definition of the group Φ' and the fact $L \subset R$, it follows at once

$$\Phi'(L) \supset \Phi'(R). \qquad (2.2)$$

Concerning the EULER characteristic $\chi(R/L)$ and the orders $o(L)$, $o(R)$ of the finite group $\Phi'(L)$, $\Phi'(R)$, we have the formula [8, (1.1)]

$$\chi(R/L) = o(R)/o(L).$$

Our assumption $\chi(W) = \chi(R/L) = 1$ then implies that $o(R) = o(L)$, and then (2.2) tells us that

$$\Phi'(R) = \Phi'(L). \qquad (2.3)$$

Now we have two connected compact LIE groups L and R. Either of them has its own singular hyperplanes in the same euclidean space E^l. From the fact that $L \subset R$ we can see immediately that a singular element of L is also a singular element of R. Thus singular hyperplanes of L are also singular hyperplanes of R. Let us denote, respectively, by

$$\pi_1, \pi_2, \ldots, \pi_m$$

and

$$\pi_1, \pi_2, \ldots, \pi_m, \theta_1, \theta_2, \ldots, \theta_h,$$

[2]) A group is said to have no centre if it possesses no other centre than the identity.

the singular hyperplanes of L and R which pass through the origin. Then from (1.1), it follows that

$$r' = 2m + l, \; r = 2(m+h) + l \quad . \quad . \quad . \quad . \quad . \quad (2.4)$$

where $r' = \dim. L$, $r = \dim. R$.

We are going to show $h = 0$. Suppose that $h \neq 0$. Then there exists a singular hyperplane θ_1 of R which is not a singular hyperplane of L and which passes through the origin. Choose a point P on θ_1 such that it does not lie on any of the hyperplanes $\pi_1, \pi_2, \ldots, \pi_m$. Since R is *semi-simple*, $\Phi'(R)$ contains the reflection φ' about the singular hyperplane θ_1 [7, p. 364]. On account of (2.3), φ' also belongs to $\Phi'(L)$. This φ' evidently differs from the identity transformation, and moreover it leaves the point P invariant. Proposition (1.2) then assures the existence of a singular hyperplane of L passing through both P and the origin. However, by the choice of P, none of the π's passes through P. This leads to a contradiction. Hence $h = 0$ and (2.4) implies $r = r'$. In other words, R and L have the same dimension. As R is connected, L and R coincide. Thus $W = R/L$ consists of only one point. This proves Theorem I.

3. Our main theorem can be stated as follows:

Theorem II. *Let W be a simply-connected manifold with* EULER *characteristic equal to two. If W admits, effectively and transitively, a compact connected group R, then W is a sphere of even dimension, and R is either the orthogonal group or the exceptional simple* LIE *group G_2 of* CARTAN*'s class G and rank 2. G_2 presents only when W is six-dimensional.*

In order to prove this theorem, we shall first establish a series of lemmas.

Lemma 1. *Let W be a simply-connected manifold with* EULER *characteristic equal to a prime number p. Then the connected compact group R which can possibly act on W transitively and effectively must be a simple* LIE *group.*

Proof. Let L denote the isotropic subgroup of R. Then W can be regarded as the coset space R/L. Since $\chi(W) = p \neq 0$, R has no centre [8, (2.1)]. If R is not simple, then R can be expressed as the direct product $R' \times R''$ of two connected compact semi-simple LIE groups none of which consists of only one element [8, (2.2)]. Then W is homeomorphic to the topological product

$$(R'/L') \times (R''/L'')$$

of two coset spaces where $L' = R' \cap L$, $L'' = R'' \cap L$ [8, (2.3)]. From the well-known KÜNNETH's formula, we have

$$p = \chi(W) = \chi(R'/L') \cdot \chi(R''/L'').$$

However, p is a prime number so that one of the factors in the right hand side of the above equality must be unity. We can assume that $\chi(R'/L') = 1$. Since W is simply-connected, R'/L' must be simply-connected as well.

Moreover, R'/L' admits transitively the compact LIE group R'. From Theorem I, R'/L' is a single point. Hence $R' = L'$.

Now we return to our original homogeneous space $W = R/L$. Since R' is an invariant subgroup of R and $R' = L' \subset L$, it follows that each element of R' leaves unaltered every point of W. R' has more than one elements. This contradicts our assumption that R is effective on W. Hence R is simple.

Lemma 2. *Let R_1 and R_2 be two locally isomorphic connected compact* LIE *groups. Then $\Phi(R_1) \approx \Phi(R_2)$.*

Proof. If R_1 and R_2 are semi-simple, this lemma is well-known. In the general case, we know that there exists a connected compact LIE group R such that [6, Theorem 87]

$$R_1 = R/N_1, \qquad R_2 = R/N_2$$

where N_1 and N_2 are discrete subgroups of R belonging to the centre. Hence we have two natural homomorphic onto-mappings

$$f_1 : R \to R_1, \quad f_2 : R \to R_2$$

with kernels N_1 and N_2 respectively. Let T be a maximal total subgroup of R. N_1 and N_2 being contained in the centre of R, are contained in T. From HOPF's result [3, 1.6], it follows that

$$\Phi(R) \approx \Phi(R_1), \quad \Phi(R) \approx \Phi(R_2),$$

and hence $\Phi(R_1) \approx \Phi(R_2)$.

Lemma 3. *Let L be a connected compact* LIE *group locally isomorphic to the direct product*

$$L_1 \times L_2 \times \dots \times L_s$$

of connected compact LIE *groups L_j $(j = 1, 2, \dots, s)$. Then*

$$o(L) = o(L_1) \cdot o(L_2) \dots o(L_s).$$

Proof. This is a direct consequence of Lemma 2 and [8, (14.1)].

4. In this section and the next, we shall study some properties of the CARTAN's exceptional groups.

Lemma 4. *The* CARTAN*'s exceptional group F_4, E_6, E_7, E_8 cannot act transitively on a simply-connected manifold with* EULER *characteristic equal to two.*

Proof. Let us first consider the case E_8. Suppose W to be a simply-connected manifold with EULER characteristic equal to two. If there exists an E_8 acting on W transitively, then the universal covering group $\widetilde{E}_8$ of E_8 acts on W in the natural manner. This group $\widetilde{E}_8$ is simply-connected and is also of the class E_8. Let L be the isotropic of subgroup $\widetilde{E}_8$. We can regard W as the coset space $\widetilde{E}_8/L$. From the simply-connectedness of both W and $\widetilde{E}_8$, it follows that L is connected [1, § 31]. Therefore [8, (1.1)]

$$2 = \chi(W) = \chi(\widetilde{E}_8/L) = o(\widetilde{E}_8)/o(L).$$

The table (1.3) tells us that $o(\widetilde{E}_8) = 10! \cdot 3 \cdot 2^6$. Hence we have

$$o(L) = 10! \cdot 3 \cdot 2^5. \quad \ldots \ldots \quad (4.1)$$

The isotropic subgroup L, being closed in $\widetilde{E}_8$, form a compact LIE group. Hence it is locally isomorphic with a direct product of the form

$$L_1 \times L_2 \times \ldots \times L_s \quad \ldots \ldots \quad (4.2)$$

where L_j is either a toral group or a compact simple LIE group. Let us denote by l_j the rank of L_j. Then

$$l_1 + l_2 + \ldots + l_s = \text{rank of } L.$$

Since $\chi(\widetilde{E}_8/L) \neq 0$, L has the same rank 8 as the group $\widetilde{E}_8$, and hence

$$l_1 + l_2 + \ldots + l_s = 8. \quad \ldots \ldots \quad (4.3)$$

Furthermore, we have from Lemma 3 that $o(L) = o(L_1) \cdot o(L_2) \ldots o(L_s)$. Equality (4.1) then implies

$$o(L_1) \cdot o(L_2) \ldots o(L_s) = 10! \, 3 \cdot 2^5. \quad \ldots \ldots \quad (4.4)$$

Evidently L_j cannot be an E_8, and from (4.3) it follows that $l_j \leq 8$. Moreover, one of the $o(L_j)$ must be divisible by 7. For definiteness, let it be $o(L_1)$. Table (1.3) then tells us that L_1 is one of the following groups

$$A_8, A_7, A_6, B_8, B_7, C_8, C_7, D_8, D_7, E_7.$$

Hence $o(L_1)$ is not divisible by 25. One of the factors $o(L_2), \ldots, o(L_s)$ must be divisible by 5. However, $l_j \leq 8 - l_1 \leq 2$ $(j = 2, 3, \ldots, s)$. It follows from (1.3) that this is impossible.

From the above discussions, we know that an E_8 cannot act transitively on W. Thus the lemma is proved for the case E_8. The other cases can be treated similarly. Q. E. D.

Lemma 5. *Let W be a simply-connected space with* EULER *characteristic equal to two. If it admits transitively the exceptional group G_2, then the isotropic subgroup must be an A_2.*

P r o o f. Let L denote the isotropic subgroup. By similar reasoning as above, we know that L is of rank 2 and $o(L) = 6$. (1.3) and Lemma 3 tell us that L is an A_2.

5. In the preceding section, we know that there is possibly a simply-connected homogeneous space of G_2 with EULER characteristic equal to two. Now we shall show that this space is the sphere of six dimension.

Lemma 6. *Up to an automorphism of G_2, there is one and only one subgroup L of G_2 such that L is an A_2.*

P r o o f. Noting the fact that G_2 is simply-connected [7, p. 378], we can prove this lemma by reasoning analogous to, though far simpler than, that the author used in [8, Part II]. The details are omitted.

Lemma 7. *The six-dimensional sphere S^6 is a homogeneous space of the group G_2, and is the only simply-connected homogeneous space of G_2 which has* EULER *characteristic equal to two.*

P r o o f. The uniqueness follows directly from Lemma 6 and Lemma 7. We need only prove the first part of our lemma. It is well-known that G_2 can be embedded in the group Γ_6 of all proper orthogonal matrices with 7 rows and columns. Therefore, we can assume $G_2 \subset \Gamma_6$.

Let S^6 be the unit sphere in the seven-dimensional euclidean space. Γ_6 acts transitively and effectively on S^6. G_2, being a subgroup of Γ_6, acts on S^6 in the natural manner. We shall show that G_2 is transitive on S^6. For this purpose, let us consider the orbits of G_2 [10, p. 194]. Let m be the maximum of the dimension of all the orbits. There is a point x of S^6 such that the orbit $G_2(x)$ is m-dimensional. Since each orbit is connected and G_2 acts effectively on S^6, we have

$$0 < m \leqslant 6 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5.1)$$

The group G_2 acts transitively on $G_2(x)$ so that $G_2(x)$ can be regarded as a coset space G_2/L where L is a closed subgroup of G_2. Since G_2 is a simple LIE group without centre [7, p. 378], it has no proper invariant subgroup. Furthermore,

$$\text{dim. } G_2 - \text{dim. } L = \text{dim. } G_2(x) = m > 0.$$

Thus $L \neq G_2$ and thus G_2 acts effectively on $G_2(x)$ [1, § 18]. It follows then [10, p. 202]

$$14 = \text{dim. } G_2 \leqslant m\,(m+1)/2. \; . \; . \; . \; . \; . \; . \quad (5.2)$$

Combining (5.1) and (5.2), we know that m is either equal to 5 or equal to 6.

Suppose $m = 5$. Then [5, p. 465] all orbits of G_2 are 5-dimensional except for two orbits of lower dimension. Let these two exceptional orbits be $G_2(y_1)$ and $G_2(y_2)$ $(y_1, y_2 \,\varepsilon\, S^6)$. Then $G_2(y_i)$ must be of zero dimension, for otherwise by using similar reasoning as above we can show that its dimension is either 5 or 6. Hence $G_2(y_i) = y_i$. In other words, y_i $(i = 1, 2)$ is a fixed point of G_2. All the matrices of Γ_6 leaving y_1 unaltered form an orthogonal group Γ_5 of order 6. Hence

$$G_2 \supset \Gamma_5. \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (5.3)$$

However, Γ_5 cannot have any proper subgroup of dimension greater than ten [5, p. 463]. Hence (5.3) is absurd. The integer m cannot be equal to 5, and therefore $m = 6$. It follows then that $G_2(x) = S^6$. In other words, S^6 is really a homogeneous space of G_2. This completes the proof.

6. In a recent paper [8], the author has determined all the spaces which have non-vanishing EULER characteristic and admit transitively a classical

group (i.e., simple group of the four main classes). According as the group is an A_k, B_k, C_k, or D_k, the space is called an elementary χ_A-, χ_B-, χ_C- or χ_D-space respectively, and we denote it by W_A, W_B, W_C or W_D. To each W_A (W_B, W_C, W_D) is associated a set Θ_A (Θ_B, Θ_C, Θ_D) of integers, called category [8, § 9], given as follows:

$$\Theta_A = \{e;\, a_1, a_2, \ldots, a_n\}, \qquad (6.1A)$$

$$\Theta_B = \{e;\, a_1, a_2, \ldots, a_n;\, d_1, d_2, \ldots, d_{m'};\, b\}, \qquad (6.1B)$$

$$\Theta_C = \{e;\, a_1, a_2, \ldots, a_u;\, c_1, c_2, \ldots, c_m\}. \qquad (6.1C)$$

$$\Theta_D = \{e;\, a_1, a_2, \ldots, a_n;\, d_1, d_2, \ldots, d_m\}. \qquad (6.1D)$$

These categories Θ_A, Θ_B, Θ_C, Θ_D satisfy, respectively, the following conditions

$$e \geqq n-1,\ a_\alpha \geqq 2,\ e + \Sigma a_\alpha = l, \qquad (6.2A)$$

$$e \geqq n,\ a_\alpha \geqq 2.\ d_\beta \geqq 2,\ b \geqq 0,\ e - n + \Sigma a_\alpha + \Sigma d_\beta = l, \qquad (6.2B)$$

$$e \geqq n,\ a_\alpha \geqq 2,\ c_\beta \geqq 1,\ e - n + \Sigma a_\alpha + \Sigma c_\beta = l, \qquad (6.2C)$$

$$e \geqq n,\ a_\alpha \geqq 2,\ d_\beta \geqq 2,\ e - n + \Sigma a_\alpha + \Sigma d_\beta = l. \qquad (6.2D)$$

where l denotes the rank of the group which the space admits. It has been shown that [8, § 13, Corollary 6]

(6.3) *Two simply-connected elementary χ-spaces with the same category are homeomorphic.*

Let W_A, W_B, W_C and W_D be *simply-connected* elementary χ-spaces of category Θ_A, Θ_B, Θ_C and Θ_D respectively. Concerning their EULER characteristic χ, we have the formulae

$$\left.\begin{aligned}
\chi(W_A) &= \frac{(l+1)!}{a_1!\, a_2! \ldots a_n!} \\
\chi(W_B) &= \frac{2^{(l+m'-b-d_1-d_2-\ldots-d_{m'})}\, l!}{a_1! \ldots a_n!\, d_1! \ldots d_{m'}!\, b!} \\
\chi(W_C) &= \frac{2^{(-c_1-c_2-\ldots-c_m)}\, l!}{a_1! \ldots a_n!\, c_1! \ldots c_m!} \\
\chi(W_D) &= \frac{2^{(l+m-d_1-d_2-\ldots-d_m-1)}\, l!}{a_1! \ldots a_n!\, d_1! \ldots d_m!}
\end{aligned}\right\} \qquad (6.4)$$

From (6.2) and (6.4), it is easy to single out those spaces whose EULER characteristic is equal to two. In fact, we have

Lemma 8. *Let W be a simply-connected space which has* EULER *characteristic equal to two and admits transitively and effectively a classical group R. Then W is homeomorphic with a sphere of even dimension, and R is the orthogonal group.*

Proof. Since R is a classical group and $\chi(W) \neq 0$, W is an elementary χ_A-, χ_B-, χ_C- or χ_D-space. Bearing in mind the simply-connectedness of W, we can easily see from (6.2) and (6.4) that R must be one of the following groups

$$A_1, C_1, C_2, B^l \ (l = 1, 2, \ldots).$$

However, it is well-known that $A_1 = C_1 = B_1$, $C_2 = B_2$ so that R is an B_l and W is an elementary χ_B-space. By hypothesis, R acts on W effectively. Hence R has no centre [8, (2.1)] and hence R is the group of all proper orthogonal matrices with $2l + 1$ rows and columns. Moreover, (6.4) tells us that the category Θ_B of the space $W = B_l/L$ is specified as follows

$$e = b = n = 0, \ m' = 1, \ d_1 = l. \quad \ldots \ldots \quad (6.5)$$

On the other hand, the $2l$-dimensional sphere S^{2l} is an elementary χ_B-space B_l/D_l. From the very definition of category, it follows that the category of S^{2l} is also given by (6.5). On account of (6.3), W and S^{2l} are homeomorphic. The lemma is thus proved.

7. Proof of Theorem II. Let W be a simply-connected manifold admitting effectively and transitively a connected compact group R, and having EULER characteristic equal to two. From Lemma 1, we know that R is a connected compact simple LIE group. Lemma 4 tells us that R cannot be of the classes F_4, E_6, E_7, E_8. Hence R is either the exceptional group G_2 or a classical group.

If R is the group G_2, it follows from Lemma 7 that W is the six-dimensional sphere. In the other alternative, Lemma 8 tells us that W is a sphere of even dimension and R the orthogonal group. Theorem II is therefore proved.

On account of Lemmas 5 and 7, we know that S^6 is homeomorphic with the coset space G_2/A_2. From this fact and the well-known homotopy sequence, it follows immediately

Let $\pi_n(G_2)$ denote the n^{th} homotopy group of the exceptional group G_2. Then $\pi_3(G_2)$ is free cyclic and

$$\pi_1(G_2) = \pi_2(G_2) = \pi_4(G_2) = 0.$$

Furthermore, by using similar method as in the proof of Theorem II we can prove the following

Theorem III [3]. *Let W be a simply-connected manifold with* EULER *characteristic equal to a prime number $p > 2$. If it admits transitively a compact group of transformations. Then W is either a complex projective space of $2(p-1)$ dimension, or a quaternion projective space of $4(p-1)$*

[3]) From a recent personal correspondence, the author learnt that some of these result are known to A. BOREL.

dimension or a 16-dimensional closed orientable manifold with POINCARE *polynomial*

$$1 + t^8 + t^{16}.$$

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REFERENCES.

1. S. CARTAN, La théorie des groupes finis et continus et l'analysis situs (Mémorial Sc. Math., XLII), 1930 Paris.
2. H. HOPF, Ueber den Rang geschlossener Liescher Gruppen, Comment. Math. Helvet., **13**, 119—143 (1941).
3. ———, Maximale Toroide und singuläre Elemente in geschlossener Lieschen Gruppen, Ibid., **15**, 49—70 (1942).
4. H. HOPF und H. SAMELSON, Ein Satz über die Wirkungsräume geschlossener Lieschen Gruppen, Ibid., **13**, 240—251 (1941).
5. D. MONTGOMERY and H. SAMELSON, Transformation groups of spheres, Ann. of Math., **44**, 454—470 (1943).
6. L. PONTRJAGIN, Topological groups, 1939 Princeton.
7. E. STIEFEL, Ueber eine Beziehung zwischen geschlossenen Lieschen Gruppen und diskontinuierlichen Bewegungsgruppen euklidischer Räume und ihre Anwendung auf Aufzählung der einfachen Lieschen Gruppen, Comment. Math. Helvet., **14**, 350—380 (1941).
8. HSIEN-CHUNG WANG, Homogeneous space with non-vanishing Euler characteristic, To appear in Ann. of Math., No. 3, Vol. 50 (1949).
9. E. WITT, Spiegelungsgruppen und Aufzählung halbeinfacher Liescher Ringe, Hamb. Abh., **14**, 289—322 (1941).
10. L. ZIPPIN, Transformation groups, Lectures in Topology, 1941 Michigan, 191—221.