Mathematics. — *On the Geometry of a System of Partial Differential Equations of the Second Order.* By A. URBAN. (Communicated by Prof. J. A. SCHOUTEN.)

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1. The geometry of the system of partial differential equations of the second order

$$\partial^2_{\mu\lambda} z^h = H^h_{\mu\lambda}(\xi^\varkappa, z^j, \partial_\nu z^k); \quad \varkappa, \lambda, \mu, \nu = 1, \dots, n; \quad h, i, j, k = 1, \dots, \mathrm{m}, \quad (1.1)$$

and similar systems of higher order with the unknown functions z^h and the independent variables $\xi^\varkappa$ was treated already many times. An extensive bibliography of the subject is found in E. BORTOLOTTI [3] [1]). The usual geometrical interpretation of the system (1.1) may be characterized as follows. If the invariance of the equations (1.1) for certain transformations in the variables $\xi^\varkappa$ or $\xi^\varkappa, z^h$ is established and if from the $H^h_{\mu\lambda}$ the components of some linear connexion and (or) some other geometric objects in the space of the $\xi^\varkappa$ or $\xi^\varkappa$, z^h can be derived such that vice versa the $H^h_{\mu\lambda}$ can be derived from the components, the geometry in this space established by these objects gives the geometric interpretation of the given system of equations.

All this is very simple for $m = 1$ and for $H^1_{\mu\lambda}$ linear in $\partial_\lambda z$ and z

$$\partial^2_{\mu\lambda} z = \Gamma^\varkappa_{\mu\lambda} \partial_\varkappa z + \Gamma_{\mu\lambda} z; \quad \varkappa, \lambda, \mu = 1, \dots, n. \quad . \quad . \quad . \quad (1.2)$$

For the transformations

$$\xi^{\varkappa'} = \xi^{\varkappa'}(\xi^\varkappa); \quad \varkappa = 1, \dots, n; \quad \varkappa' = 1', \dots, n', \quad . \quad . \quad . \quad (1.3)$$

the functions $\Gamma^\varkappa_{\mu\lambda}$ resp. $\Gamma_{\mu\lambda}$ transform as the components of a connexion and a tensor resp. [2]). Therefore a geometry of the system (1.2) is the geometry of a space X_n with the connexion $\Gamma^\varkappa_{\mu\lambda}$ and the tensor $\Gamma_{\mu\lambda}$. All this is much more complicated for the analogical systems of higher orders. E. BOMPIANI [3]) studied the system of the third order.

A. MAXIA [4]) gave another geometrical interpretation of the system (1.2). If the z^a; $a = 0, 1, \dots, n$, are affine coordinates in an affine space E_{n+1}, the $n + 1$ solutions $z^a = z^a(\xi^\varkappa)$ of the system (1.2) represent a hypersurface A_n in E_{n+1} for which the system (1.2) together with $\partial_\lambda z = z_\lambda$ may be considered as a fundamental system. For this A_n, $\Gamma_{\mu\lambda}$ is the fundamental tensor and the $\Gamma^\varkappa_{\mu\lambda}$'s are the coefficients of the induced connexion. A. MAXIA proved that the same hypersurface A_n considered as

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1) The numbers in [ ] refer to the bibliography at the end of this paper.

2) Cf. [1].

3) [2].

4) [5].

an envelope of its tangent-E_n's is represented by the conjugated system of the system (1. 2) introduced by L. BIANCHI [5]).

In the present paper still another geometrical interpretation of the system (1. 2) is given. This system is very closely connected with the system (3. 2). Every solution of (3. 2) may be considered as a geodesic hypersurface of a space X_{n+1} with the projective connexion (3. 15). In the sections **1—6** this geometrical interpretation is developed.

The most general transformation which leaves the form of the system (1. 2) unchanged is given by (2. 8). § **7** is closely connected with the investigations of A. MAXIA and deals more in detail with the functions $\Gamma^{\varkappa}_{\mu\lambda}$ and $\Gamma_{\mu\lambda}$.

In the § **8** the conjugated system of L. BIANCHI of the system (1. 2) is constructed by means of geometrical considerations.

2. Let be given a completely integrable system of linear partial differential equations of the second order

$$\partial^2_{\mu\lambda} z = \Gamma^{\varkappa}_{\mu\lambda} \partial_{\varkappa} z + \Gamma_{\mu\lambda} z\,; \quad \varkappa, \lambda, \mu, \nu, \varrho = 1, \ldots, n, \quad . \quad . \quad (2.1)$$

in the unknown variable z and the independent variables $\xi^{\varkappa}$. We suppose the coefficients $\Gamma^{\varkappa}_{\mu\lambda} = \Gamma^{\varkappa}_{\lambda\mu}$, $\Gamma_{\mu\lambda} = \Gamma_{\lambda\mu}$ to be analytic functions of the $\xi^{\varkappa}$ only. According to the assumption that the system (2. 1) is completely integrable, these coefficients are related by the following equations

$$\left.\begin{aligned} &\text{a)} \quad \partial_{[\nu} \Gamma^{\varkappa}_{\mu]\lambda} + \Gamma^{\varkappa}_{\varrho[\nu} \Gamma^{\varrho}_{\mu]\lambda} + A^{\varkappa}_{[\nu} \Gamma_{\mu]\lambda} = 0, \\ &\text{b)} \quad \partial_{[\nu} \Gamma_{\mu]\lambda} + \Gamma_{\varrho[\nu} \Gamma^{\varrho}_{\mu]\lambda} = 0. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2.2)$$

The system (2. 1) is invariant, i.e. it transforms into

$$\partial^2_{\mu'\lambda'} z' = \Gamma^{\varkappa'}_{\mu'\lambda'} \partial_{\varkappa'} z' + \Gamma_{\mu'\lambda'} z'\,; \quad \varkappa', \lambda', \mu', \nu', \varrho' = 1', \ldots, n' \quad . \quad (2.3)$$

if and only if the $\xi^{\varkappa'}$ depend on the $\xi^{\varkappa}$ only

$$\xi^{\varkappa'} = \xi^{\varkappa'}(\xi^{\varkappa}), \quad \text{det.} \left(\frac{\partial \xi^{\varkappa'}}{\partial \xi^{\varkappa}}\right) \neq 0 \quad . \quad . \quad . \quad . \quad . \quad (2.4)$$

and if z transforms as follows

$$z' = \varrho(\xi^{\varkappa}) z + \sigma(\xi^{\varkappa})\,; \quad \varrho(\xi^{\varkappa}) \neq 0. \quad . \quad . \quad . \quad . \quad . \quad (2.5)$$

where $\sigma(\xi^{\varkappa})$ is a solution of (2. 3). The relations between the coefficients in (2. 1) and (2. 3) are given by

$$\Gamma^{\varkappa'}_{\mu'\lambda'} = \Gamma^{\varkappa}_{\mu\lambda} A^{\mu\lambda\varkappa'}_{\mu'\lambda'\varkappa} + A^{\varkappa'}_{\varkappa} \partial_{\mu'} A^{\varkappa}_{\lambda'} + 2 A^{\varkappa'}_{(\mu'} \partial_{\lambda')} \log \varrho \quad . \quad . \quad (2.6)$$

and

$$\left.\begin{aligned} \Gamma_{\mu'\lambda'} = \Gamma_{\mu\lambda} A^{\mu\lambda}_{\mu'\lambda'} - \Gamma^{\varkappa}_{\mu\lambda} A^{\mu\lambda\varkappa'}_{\mu'\lambda'\varkappa} \partial_{\varkappa'} \log \varrho - (\partial_{\mu'} A^{\varkappa}_{\lambda'}) A^{\varkappa'}_{\varkappa} \partial_{\varkappa'} \log \varrho \,- \\ - (\partial_{\mu'} \log \varrho)(\partial_{\lambda'} \log \varrho) + \partial^2_{\mu'\lambda'} \log \varrho\,. \end{aligned}\right\} \quad (2.7)$$

[5]) E. BOMPIANI [2] gave a similar geometrical interpretation for the analogical systems of the third order. The systems of arbitrary orders of this type may be treated in the same way.

As in (2. 5) the addition of a solution $\sigma(\xi^\varkappa)$ of the system (2. 3) may be considered as a trivial transformation, we restrict ourselves in the following to the transformations (2. 4) and (2. 5) with $\sigma = 0$,

$$\left.\begin{array}{ll}\text{a)} & \xi^{\varkappa'} = \xi^{\varkappa'}(\xi^\varkappa), \ \text{det.}\left(\dfrac{\partial \xi^{\varkappa'}}{\partial \xi^\varkappa}\right) \neq 0,\\ \text{b)} & z' = \varrho(\xi^\varkappa)\, z, \ \ \varrho(\xi^\varkappa) \neq 0.\end{array}\right\} \quad \ldots \quad (2.8)$$

3. In order to find a geometrical interpretation of the system (2. 1) let us consider a space X_{n+1} with the coordinates $z, \xi^\varkappa$. If in this space we introduce a new coordinate

$$\xi^0 \stackrel{\text{def}}{=} -\log z \quad \ldots \ldots \quad (3.1)$$

instead of z, the form of (2. 1) is not invariant and (2. 1) is transformed into

$$\partial^2_{\mu\lambda}\, \xi^0 - (\partial_\mu \xi^0)(\partial_\lambda \xi^0) = \Gamma^\varkappa_{\mu\lambda}\, \partial_\varkappa \xi^0 - \Gamma_{\mu\lambda} \quad \ldots \ldots \quad (3.2)$$

It is evident that by means of (3. 1) to every solution of (2. 1) with the exception of the trivial solution $z = 0$ there corresponds one and only one solution of (3. 2) and conversely. Hence also (3. 2) is completely integrable if (2. 2) holds.

$$\left.\begin{array}{ll}\text{a)} & \xi^{0'} = \xi^0 - \log \varrho(\xi^\varkappa), \ \xi^{\varkappa'} = \xi^{\varkappa'}(\xi^\varkappa); \ \text{det.}\left(\dfrac{\partial \xi^{\varkappa'}}{\partial \xi^\varkappa}\right) \neq 0,\\ \text{b)} & \xi^0 = \xi^{0'} + \log \varrho(\xi^\varkappa(\xi^{\varkappa'})), \ \xi^\varkappa = \xi^\varkappa(\xi^{\varkappa'}); \ \varrho(\xi^\varkappa) \neq 0,\\ & \qquad \varkappa = 1, \ldots, n; \ \varkappa' = 1', \ldots, n',\end{array}\right\} \quad . \quad (3.3)$$

corresponding to (2. 8) and leaving (3. 2) invariant are in the space X_{n+1} with the coordinates ξ^α; $\alpha, \beta, \gamma, \delta = 0, 1, \ldots, n$, the allowable coordinate transformations. Hence the X_{n+1} of $\xi^0, \xi^\varkappa$ is not an ordinary X_{n+1} but an X_{n+1} with a restricted group of coordinate transformations.

For the transformations (3. 3) $\Gamma^\varkappa_{\mu\lambda}$ and $\Gamma_{\mu\lambda}$ transform according to (2. 6, 7).

The mixed components of the unity affinor are

$$\left.\begin{array}{l} A^{0'}_0 = \dfrac{\partial \xi^{0'}}{\partial \xi^0} = 1, \ A^{0'}_\lambda = \dfrac{\partial \xi^{0'}}{\partial \xi^\lambda} = -\partial_\lambda \log \varrho, \ A^{\varkappa'}_0 = \dfrac{\partial \xi^{\varkappa'}}{\partial \xi^0} = 0, \ A^{\varkappa'}_\varkappa = \dfrac{\partial \xi^{\varkappa'}}{\partial \xi^\varkappa},\\ A^0_{0'} = \dfrac{\partial \xi^0}{\partial \xi^{0'}} = 1, \ A^0_{\lambda'} = \dfrac{\partial \xi^0}{\partial \xi^{\lambda'}} = \partial_{\lambda'} \log \varrho, \ A^\varkappa_{0'} = \dfrac{\partial \xi^\varkappa}{\partial \xi^{0'}} = 0, \ A^\varkappa_{\varkappa'} = \dfrac{\partial \xi^\varkappa}{\partial \xi^{\varkappa'}}.\end{array}\right\} \quad (3.4)$$

According to the fact that the group of coordinate transformations in X_{n+1} is a restricted one there exists in X_{n+1} 1° an invariant congruence of curves

$$\xi^\varkappa = \text{const.} \quad \ldots \ldots \quad (3.5)$$

(individual curves of this congruence need not be invariant) and 2° a class of individually invariant contravariant vectors characterized by the vanishing of all components except the 0-component.

Now let ${}^*\Gamma^{\alpha}_{\gamma\beta}$ be an arbitrary symmetric linear connexion given in X_{n+1}. As we may easily prove, for the transformations (3. 3) the coefficients of this connexion transform as follows

$$\left.\begin{aligned}{}^*\Gamma^{\varkappa'}_{\mu'\lambda'} = {}^*\Gamma^{\varkappa}_{\mu\lambda} A^{\mu\,\lambda\,\varkappa'}_{\mu'\lambda'\varkappa} + 2\,{}^*\Gamma^{\varkappa}_{\mu 0} A^{\varkappa'\mu}_{\varkappa(\mu'} \partial_{\lambda')} \log \varrho + &\\ + {}^*\Gamma^{\varkappa}_{00} A^{\varkappa'}_{\varkappa} (\partial_{\mu'} \log \varrho)(\partial_{\lambda'} \log \varrho) + A^{\varkappa'}_{\varkappa} \partial_{\mu'} A^{\varkappa}_{\lambda'},&\end{aligned}\right\} \quad (3.\,6)$$

$$\left.\begin{aligned}{}^*\Gamma^{0'}_{\mu'\lambda'} = {}^*\Gamma^{0}_{\mu\lambda} A^{\mu\,\lambda}_{\mu'\lambda'} + 2\,{}^*\Gamma^{0}_{\mu 0} A^{\mu}_{(\mu'} \partial_{\lambda')} \log \varrho + {}^*\Gamma^{0}_{00} (\partial_{\mu'} \log \varrho)(\partial_{\lambda'} \log \varrho) - &\\ -[{}^*\Gamma^{\varkappa}_{\mu\lambda} A^{\mu\,\lambda\,\varkappa'}_{\mu'\lambda'\varkappa} + 2\,{}^*\Gamma^{\varkappa}_{\mu 0} A^{\gamma'\mu}_{\varkappa(\mu'} \partial_{\lambda')} \log \varrho + {}^*\Gamma^{\varkappa}_{00} A^{\varkappa'}_{\varkappa} (\partial_{\mu'} \log \varrho)(\partial_{\lambda'} \log \varrho) + &\\ + A^{\varkappa'}_{\varkappa} \partial_{\mu'} A^{\varkappa}_{\lambda'}] \cdot \partial_{\varkappa'} \log \varrho + \partial^2_{\mu'\lambda'} \log \varrho,&\end{aligned}\right\} \quad (3.\,7)$$

$${}^*\Gamma^{\alpha'}_{\gamma' 0'} = {}^*\Gamma^{\alpha}_{\gamma 0} A^{\gamma\,\alpha'}_{\gamma'\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.\,8)$$

$$\varkappa, \lambda, \mu = 1, \dots, n \quad ; \quad \varkappa', \lambda', \mu' = 1', \dots, n',$$
$$\alpha, \beta, \gamma = 0, 1, \dots, n \quad ; \quad \alpha', \beta', \gamma' = 0', 1', \dots, n'.$$

In our A_{n+1} (i.e. in the X_{n+1} with the connexion ${}^*\Gamma^{\alpha}_{\gamma\beta}$) there exists a well defined set of X_n's whose equations are the solutions

$$\xi^0 = f(\xi^{\varkappa}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.\,9)$$

of the system (3. 2). The question arises whether there exist any connexions ${}^*\Gamma^{\alpha}_{\gamma\beta}$ in A_{n+1} for which all these X_n's are geodesic X_n's.

It is easily proved that (3. 9) represents a geodesic X_n in X_{n+1} if and only if the function $f(\xi^{\varkappa})$ is a solution of the system

$$\left.\begin{aligned}\partial^2_{\mu\lambda} f + ({}^*\Gamma^{0}_{00} A^{\varkappa}_{(\mu} - 2\,{}^*\Gamma^{\varkappa}_{(\mu|0|})(\partial_{\lambda)} f)(\partial_{\varkappa} f) - &\\ - {}^*\Gamma^{\varkappa}_{00} (\partial_{\mu} f)(\partial_{\lambda} f)(\partial_{\varkappa} f) - ({}^*\Gamma^{\varkappa}_{\mu\lambda} - 2\,{}^*\Gamma^{0}_{(\mu|0|} A^{\varkappa}_{\lambda)}) \partial_{\varkappa} f + {}^*\Gamma^{0}_{\mu\lambda} = 0.&\end{aligned}\right\} \quad (3.\,10)$$

Accordingly every solution of (3. 2) has to be a solution of (3. 10). But because of the complete integrability of (3. 2) this is possible if and only if (3. 2) and (3. 10) are equivalent. Hence

$$\left.\begin{aligned}&\text{a)}\quad {}^*\Gamma^{\varkappa}_{00} = 0, &&\text{b)}\quad {}^*\Gamma^{0}_{\mu\lambda} = \Gamma_{\mu\lambda},\\ &\text{c)}\quad {}^*\Gamma^{\varkappa}_{\mu\lambda} = \Gamma^{\varkappa}_{\mu\lambda} + 2\,{}^*\Gamma^{0}_{(\mu|0|} A^{\varkappa}_{\lambda)}, &&\text{d)}\quad {}^*\Gamma^{\varkappa}_{\mu 0} = \tfrac{1}{2}(1 + {}^*\Gamma^{0}_{00}) A^{\varkappa}_{\mu}.\end{aligned}\right\} \quad (3.\,11)$$

From (3. 11) we see that the connexion ${}^*\Gamma^{\alpha}_{\gamma\beta}$ is not determined uniquely. The ${}^*\Gamma^{0}_{\gamma 0}$'s may be still chosen arbitrarily but they must transform according to (3. 8) and therefore

$${}^*\Gamma^{0'}_{0'0'} = {}^*\Gamma^{0}_{00}, \quad {}^*\Gamma^{0'}_{\mu'0'} = {}^*\Gamma^{0}_{\mu 0} A^{\mu}_{\mu'} + \tfrac{1}{2}(-1 + {}^*\Gamma^{0}_{00}) \partial_{\mu'} \log \varrho. \quad (3.\,12)$$

If we write now

$$p_0 \overset{\text{def}}{=} \tfrac{1}{2}(-1 + {}^*\Gamma^{0}_{00}), \quad p_\lambda \overset{\text{def}}{=} {}^*\Gamma^{0}_{\lambda 0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.\,13)$$

we see that p_β, $\beta = 0, 1, \dots, n$, is a covariant vector. Conversely, if p_β is an arbitrary covariant vector, all connexions satisfying (3. 11) may be written in the form

$$\left.\begin{aligned} &{}^{*}\Gamma^{\varkappa}_{00}=0,\ {}^{*}\Gamma^{\varkappa}_{\mu 0}=A^{\varkappa}_{\mu}+p_0 A^{\varkappa}_{\mu},\ {}^{*}\Gamma^{\varkappa}_{\mu\lambda}=\Gamma^{\varkappa}_{\mu\lambda}+2p_{(\mu}A^{\varkappa}_{\lambda)},\\ &{}^{*}\Gamma^{0}_{00}=1+2p_0,\ {}^{*}\Gamma^{0}_{\mu 0}=p_{\mu},\ {}^{*}\Gamma^{0}_{\mu\lambda}=\Gamma_{\mu\lambda},\end{aligned}\right\} \quad (3.14)$$

or, taking all equations together

$${}^{*}\Gamma^{\alpha}_{\gamma\beta}={}^{0}\Gamma^{\alpha}_{\gamma\beta}+2p_{(\gamma}A^{\alpha}_{\beta)}\ ;\quad \alpha,\beta,\gamma=0,1,\ldots,n, \quad . \ . \ . \quad (3.15)$$

where the connexion ${}^{0}\Gamma^{\alpha}_{\gamma\beta}$ is defined by

$$\text{a)}\ \ {}^{0}\Gamma^{\varkappa}_{\mu\lambda}\overset{\text{def}}{=}\Gamma^{\varkappa}_{\mu\lambda},\qquad \text{b)}\ \ {}^{0}\Gamma^{0}_{\mu\lambda}\overset{\text{def}}{=}\Gamma_{\mu\lambda},\qquad \text{c)}\ \ {}^{0}\Gamma^{\alpha}_{\gamma 0}\overset{\text{def}}{=}A^{\alpha}_{\gamma}. \quad (3.16)$$

(3. 15) is a so called projective transformation of the connexion, i.e. a transformation leaving the set of all geodesics invariant.

Hence we have proved:

Theorem I. *A necessary and sufficient condition that every solution* $\xi^0=f(\xi^{\varkappa})$ *of* (3. 2) *represents a geodesic* X_n *in the* A_{n+1} *with the connexion* ${}^{*}\Gamma^{\alpha}_{\gamma\beta}$ *is that* (3. 11) *be satisfied. The connexion* ${}^{*}\Gamma^{\alpha}_{\gamma\beta}$ *is determined to within the projective transformation* (3. 15).

All connexions (3. 15) define a so called projective connexion. The projective curvature affinor $P_{\dot{\delta}\dot{\gamma}\dot{\beta}}{}^{\alpha}$ of this projective connexion is [6])

$$P_{\dot{\delta}\dot{\gamma}\dot{\beta}}{}^{\alpha}\overset{\text{def}}{=}{}^{*}R_{\dot{\delta}\dot{\gamma}\dot{\beta}}{}^{\alpha}-2P_{[\delta\gamma]}A^{\alpha}_{\beta}+2A^{\alpha}_{[\delta}P_{\gamma]\beta} \quad . \ . \ . \quad (3.17)$$

where

$$\left.\begin{aligned} &P_{\gamma\beta}\overset{\text{def}}{=}\frac{1}{1-(n+1)^2}\{(n+2)\,{}^{*}R_{\gamma\beta}+{}^{*}V_{\gamma\beta}\},\\ &{}^{*}R_{\gamma\beta}\overset{\text{def}}{=}{}^{*}R_{\dot{\alpha}\dot{\gamma}\dot{\beta}}{}^{\alpha},\quad {}^{*}V_{\gamma\beta}\overset{\text{def}}{=}{}^{*}R_{\dot{\gamma}\dot{\beta}\dot{\alpha}}{}^{\alpha}\end{aligned}\right\} \quad . \ . \quad (3.18)$$

and where ${}^{*}R_{\dot{\delta}\dot{\gamma}\dot{\beta}}{}^{\alpha}$ is the curvature affinor of the connexion ${}^{*}\Gamma^{\alpha}_{\gamma\beta}$. It is well known that $P_{\dot{\delta}\dot{\gamma}\dot{\beta}}{}^{\alpha}$ is independent of the choice of p_{β}. For the connexion ${}^{0}\Gamma^{\alpha}_{\gamma\beta}$ we may find easily the corresponding curvature affinor ${}^{0}R_{\dot{\delta}\dot{\gamma}\dot{\beta}}{}^{\alpha}$. All its components except

$$\left.\begin{aligned} &\text{a)}\ \ \tfrac{1}{2}\,{}^{0}R_{\dot{\nu}\dot{\mu}\dot{\lambda}}{}^{\varkappa}=\partial_{[\nu}\Gamma^{\varkappa}_{\mu]\lambda}+\Gamma^{\varkappa}_{\varrho[\nu}\Gamma^{\varrho}_{\mu]\lambda}+A^{\varkappa}_{[\nu}\Gamma_{\mu]\lambda},\\ &\text{b)}\ \ \tfrac{1}{2}\,{}^{0}R_{\dot{\nu}\dot{\mu}\dot{\lambda}}{}^{0}=\partial_{[\nu}\Gamma_{\mu]\lambda}+\Gamma_{\varrho[\nu}\Gamma^{\varrho}_{\mu]\lambda}\end{aligned}\right\} \quad . \ . \quad (3.19)$$

vanish identically. But the components (3. 19) vanish in consequence of (2. 2), hence

$${}^{0}R_{\dot{\delta}\dot{\gamma}\dot{\beta}}{}^{\alpha}=0.\ {}^{7)} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (3.20)$$

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[6]) Cf. [7], II, p. 179.

[7]) The conditions of integrability (2. 2) of the system (2. 1) can be therefore replaced by the equation (3. 20).

From this result and (3. 17) we obtain:

Theorem II. *The projective curvature affinor of the projective connexion* (3. 15) *vanishes*

$$P_{\dot{\delta}\dot{\gamma}\dot{\beta}}{}^{\alpha}=0. \qquad (3.21)$$

We remark that according to (3. 20) there exists in the X_{n+1} a coordinate system (α') such that the ${}^{0}\Gamma^{\alpha'}_{\gamma'\beta'}$ all vanish. But according to (3. 16c) the transformation $(\alpha)\rightarrow(\alpha')$ does not satisfy the conditions for allowable transformations (3. 3) and therefore it does not leave invariant the form of the equations (3. 2).

4. The connexions (3. 15) are determined by the only geometric condition that all solutions $\xi^{0}=f(\xi^{\varkappa})$ of the system (3. 2) represent geodesic X_n's of the connexion ${}^{*}\Gamma^{\alpha}_{\gamma\beta}$.

Some of the connexions (3. 15)

$${}^{*}\Gamma^{\alpha}_{\gamma\beta}={}^{0}\Gamma^{\alpha}_{\gamma\beta}+2p_{(\gamma}A^{\alpha}_{\beta)} \qquad (4.1)$$

may be distinguished by introducing an additional geometric condition. We may ask for connexions of the form (4. 1) for which the invariant congruence (3. 5)

$$\xi^{\varkappa}=\text{const.} \qquad (4.2)$$

consists of pseudoparallel geodesics. A necessary and sufficient condition is that there exists a vector field v^{a} with $v^{\varkappa}=0$ such that $\delta v^{a}=\varepsilon v^{a}$ or, according to (4. 1) and (3. 16)

$$\left.\begin{aligned}&{}^{*}\Gamma^{\varkappa}_{00}\,v^{0}\,d\xi^{0}+{}^{*}\Gamma^{\varkappa}_{\mu 0}\,v\;\;d\xi^{\mu}=(1+p_{0})\,v^{0}\,d\xi^{\varkappa}=0,\\&dv^{0}+{}^{*}\Gamma^{0}_{00}\,v^{0}\,d\xi^{0}+{}^{*}\Gamma^{0}_{\mu 0}\,v^{0}\,d\xi^{\mu}=dv^{0}+(1+2p_{0})\,v^{0}\,d\xi^{0}+v^{0}p_{\mu}\,d\xi^{\mu}=\varepsilon v^{0}.\end{aligned}\right\} \qquad (4.3)$$

Hence the curves (4. 2) are always geodesics but the only values of p_{β} for which they are pseudoparallel are those for which

$$p_{0}=-1. \qquad (4.4)$$

This condition is invariant for the transformations (3. 3).

Collecting results we have:

Theorem III. *The curves of the invariant congruence* (4. 2) *are always geodesics of the connexion* ${}^{*}\Gamma^{\alpha}_{\gamma\beta}$ *; they are pseudoparallel if and only if the invariant condition* $p_{0}=-1$ *is satisfied.*

We remark that the connexion ${}^{0}\Gamma^{\alpha}_{\gamma\beta}$ does not possess this property as ${}^{*}\Gamma^{\alpha}_{\gamma\beta}={}^{0}\Gamma^{\alpha}_{\gamma\beta}$ only for $p_{\beta}=0$ which equation is invariant for the transformations (3. 3).

In order to distinguish certain connexions from (4. 1), also other additional geometric conditions can be introduced. If $\xi^{0}=f(\xi^{\varkappa})$ is a solution of (3. 2), also $\xi^{0}=f(\xi^{\varkappa})+c$ $(c=\text{const.})$ is a solution. We remark that $\xi^{0}=\text{const.}$ is not a solution, except in the case that $\Gamma_{\mu\lambda}=0$.

All the X_n's

$$\xi^0 = f(\xi^\varkappa) + c \quad ; \quad c = \text{const.}, \qquad (4.5)$$

are always geodesic X_n's of the connexions (4. 1). The solution $\xi^0 = f(\xi^\varkappa)$ being given we may ask for the connexions of the form (4. 1) for which these X_n's are pseudoparallel.

The field

$$t_0 = -1, \quad t_\lambda = \partial_\lambda f \qquad (4.6)$$

is tangent to the X_n's (4. 5). The X_n's being geodesic we have only to consider an arbitrary displacement not lying in the n-direction of t_β, e.g. a displacement with $d\xi^0 \neq 0$, $d\xi^\varkappa = 0$. Then for this displacement we have to write out the equation

$$d\xi^\lambda \, {}^*\nabla_\lambda t_\beta = \varepsilon t_\beta \qquad (4.7)$$

where ε is an arbitrary scalar (${}^*\nabla_\lambda$ symbolizis the covariant derivation with respect to ${}^*\Gamma^\alpha_{\gamma\beta}$). According to (4. 1) and (3. 16) this gives immediately

$$p_\lambda + p_0 \partial_\lambda f = 0. \qquad (4.8)$$

The general solution of this equation is

$$p_0 = -\sigma, \quad p_\lambda = \sigma \partial_\lambda f \qquad (4.9)$$

where σ is an arbitrary scalar. Hence we have proved:

Theorem IV. *The arbitrarily but definitely chosen solutions of* (3. 2)

$$\xi^0 = f(\xi^\varkappa) + c \quad ; \quad c = \text{const.}, \qquad (4.10)$$

represent always geodesic X_n's of the connexion (4. 1). *These X_n's are pseudoparallel if and only if*

$$p_0 = -\sigma, \quad p_\lambda = \sigma \partial_\lambda f \qquad (4.11)$$

where σ is an arbitrary scalar.

From the last two theorems follows immediately:

Theorem V. *If*

$$\xi^0 = f(\xi^\varkappa) + c \quad ; \quad c = \text{const.}, \qquad (4.12)$$

are ∞^1 arbitrarily but definitely chosen solutions of (3. 2), *the ∞^1 X_n's represented by* (4. 12) *are always geodesic for connexions of the form* (4. 1). *There exists one and only one among these connexions such that these X_n's are pseudoparallel and that simultaneously the curves of the invariant congruence of geodesics* (4. 2) *are pseudoparallel. This connexion is given by* (4. 1) *and*

$$p_0 = -1, \quad p_\lambda = \partial_\lambda f. \qquad (4.13)$$

5. The equation (3. 20) shows that the curvature affinor ${}^0R_{\delta\gamma\beta}^{\cdot\cdot\cdot\alpha}$ of the connexion ${}^0\Gamma^\alpha_{\gamma\beta}$ vanishes; the space A_{n+1} with this connexion is therefore an E_{n+1}. In order to find all the connexions ${}^*\Gamma^\alpha_{\gamma\beta}$ the curvature affinor of which vanishes we consider the transformation formula of the curvature affinor for the transformation (3. 15). According to ${}^0R_{\dot\delta\dot\gamma\dot\beta}{}^{\alpha} = 0$ we obtain

$$ {}^*R_{\dot\delta\dot\gamma\dot\beta}{}^{\alpha} = -2p_{[\delta\gamma]}A^{\alpha}_{\beta} + 2A^{\alpha}_{[\delta}p_{\gamma]\beta} \quad . \quad . \quad . \quad . \quad . \quad (5.\ 1) $$

where

$$ p_{\gamma\beta} \overset{\text{def}}{=} -{}^0\nabla_{\gamma}p_{\beta} + p_{\gamma}p_{\beta}, \quad ^{8)} \quad . \quad . \quad . \quad . \quad . \quad (5.\ 2) $$

${}^0\nabla_{\beta}$ being the symbol of the covariant derivation with respect to ${}^0\Gamma^{\alpha}_{\gamma\beta}$. A necessary and sufficient condition that ${}^*R_{\dot\delta\dot\gamma\dot\beta}{}^{\alpha} = 0$ is that p_{β} is a solution of the system of differential equations

$$ {}^0\nabla_{\gamma}p_{\beta} - p_{\gamma}p_{\beta} = 0. \quad ^{9)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.\ 3) $$

According to (3. 16) this system may be written as follows

$$ \left. \begin{array}{ll} \text{a)}\ \partial_0 p_0 - p_0 - p_0 p_0 = 0, & \text{b)}\ \partial_{\lambda} p_0 - p_{\lambda} - p_{\lambda} p_0 = 0, \\ \text{c)}\ \partial_0 p_{\lambda} - p_{\lambda} - p_{\lambda} p_0 = 0, & \text{d)}\ \partial_{\mu} p_{\lambda} - p_{\mu} p_{\lambda} = \Gamma^{\varkappa}_{\mu\lambda} p_{\varkappa} + \Gamma_{\mu\lambda} p_0. \end{array} \right\} \quad (5.\ 4) $$

First we consider the solutions p_{β} for which $p_0 = \text{const}$. We may easily verify that all such solutions are given by

$$ \left. \begin{array}{l} \text{a)}\ p_0 = 0, \quad p_{\lambda} = 0, \\ \text{b)}\ p_0 = -1, \quad p_{\lambda} = p_{\lambda}(\xi^{\varkappa}), \quad \partial_{\mu} p_{\lambda} - p_{\mu} p_{\lambda} = \Gamma^{\varkappa}_{\mu\lambda} p_{\varkappa} - \Gamma_{\mu\lambda}. \end{array} \right\} \quad (5.\ 5) $$

Hence we have proved:

Theorem VI. *The connexions* ${}^*\Gamma^{\alpha}_{\gamma\beta}$ *from* (4. 1), *the curvature affinor* ${}^*R_{\dot\delta\dot\gamma\dot\beta}{}^{\alpha}$ *of which vanishes and for which* $p_0 = \text{const.}$, *are either* ${}^0\Gamma^{\alpha}_{\gamma\beta}$ *or the connexions for which the congruence*

$$ \xi^{\varkappa} = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.\ 6) $$

consists of pseudoparallel geodesics.

Now let us suppose $p_0 \neq 0$. According to (5. 5) this is equivalent to ${}^*\Gamma^{\alpha}_{\gamma\beta} \neq {}^0\Gamma^{\alpha}_{\gamma\beta}$. In this case we may replace the system (5. 4) by

$$ \left. \begin{array}{l} \text{a)}\ \partial_0 p_0 - p_0 - p_0 p_0 = 0; \quad p_0 \neq 0, \\ \text{b)}\ \partial_{\lambda} p_0 - p_{\lambda} - p_{\lambda} p_0 = 0, \\ \text{c)}\ \partial_0 p_{\lambda} - p_{\lambda} - p_{\lambda} p_0 = 0, \\ \text{d)}\ \partial_{\mu} P_{\lambda} - P_{\mu} P_{\lambda} = \Gamma^{\varkappa}_{\mu\lambda} P_{\varkappa} - \Gamma_{\mu\lambda} \quad ; \quad P_{\lambda} \overset{\text{def}}{=} -\dfrac{p_{\lambda}}{p_0}. \end{array} \right\} \quad . \quad . \quad (5.\ 7) $$

Hence:

Theorem VII. *A necessary and sufficient condition that the curvature affinor* ${}^*R_{\dot\delta\dot\gamma\dot\beta}{}^{\alpha}$ *of a connexion* ${}^*\Gamma^{\alpha}_{\gamma\beta}$ $(\neq {}^0\Gamma^{\alpha}_{\gamma\beta})$ *from* (4. 1) *vanishes is that* p_{β} *is a solution of* (5. 7).

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8) Cf. [7], II, p. 177.

9) Cf. [7], II, p. 182.

6. The solutions of the system (3. 2) represent geodesic X_n's in the A_{n+1} with the connexion (4. 1). Conversely, according to the complete integrability of (3. 2) every geodesic X_n of this connexion with an equation of the form (3. 9), i.e. every geodesic X_n that has nowhere a direction in common with the invariant congruence (3. 5)

$$\xi^{\varkappa} = \text{const.} \quad \ldots \ldots \ldots \quad (6.\ 1)$$

represents a solution of (3. 2). Every other geodesic X_n consists of ∞^{n-1} curves of this congruence and its equation has the form

$$f(\xi^{\varkappa}) = 0 \quad \ldots \ldots \ldots \quad (6.\ 2)$$

with the condition

$$d\xi^{\gamma}\ {}^{*}\nabla_{\lambda}\, \partial_{\beta} f = \sigma\, \partial_{\beta} f \quad \ldots \ldots \ldots \quad (6.\ 3)$$

for every direction $d\,\xi^{a}$ for which

$$d\xi^{\varkappa}\, \partial_{\varkappa} f = 0\,, \quad d\xi_0 \text{ arbitrary}, \quad \ldots \ldots \quad (6.\ 4)$$

σ being an arbitrary scalar. From (6. 3) it follows that

$$d\,\xi^{\mu}\,(\partial^2_{\mu\lambda} f - {}^{*}\Gamma^{\varkappa}_{\mu\lambda}\, \partial_{\varkappa} f) = \sigma'\, \partial_{\lambda} f \quad \ldots \ldots \quad (6.\ 5)$$

where σ' is an arbitrary scalar.

Because of (6. 4) the equation (6. 5) is equivalent to

$$\partial^2_{\mu\lambda} f = ({}^{*}\Gamma^{\varkappa}_{\mu\lambda} + 2\,{}^{*}a_{(\mu} A^{\varkappa}_{\lambda)})\, \partial_{\varkappa} f \quad \ldots \ldots \quad (6.\ 6)$$

or in another form

$$\partial^2_{\mu\lambda} f = (\Gamma^{\varkappa}_{\mu\lambda} + 2\, a_{(\mu} A^{\varkappa}_{\lambda)})\, \partial_{\varkappa} f \quad \ldots \ldots \quad (6.\ 7)$$

where a_{μ} is an arbitrary vector and

$$a_{\lambda} = {}^{*}a_{\lambda} + p_{\lambda}\,. \quad \ldots \ldots \ldots \quad (6.\ 8)$$

The conditions of integrability of the system (6. 7) are satisfied if and only if a_{λ} is a gradient of some scalar a

$$a_{\lambda} = \partial_{\lambda}\, a\,. \quad \ldots \ldots \ldots \quad (6.\ 9)$$

That proves:

Theorem VIII. *If $f = f(\xi^{\varkappa})$ is a solution of the equations*

$$\partial^2_{\mu\lambda} f = (\Gamma^{\varkappa}_{\mu\lambda} + 2\, a_{(\mu} A^{\varkappa}_{\lambda)})\, \partial_{\varkappa} f \quad \ldots \ldots \quad (6.\ 10)$$

where $a_{\mu} = \partial_{\mu}\, a$ and a an arbitrary scalar, then $f = 0$ is the equation of a geodesic X_n for every connexion (3. 15).

7. In this section we consider the functions $\Gamma^{\varkappa}_{\mu\lambda}$ and $\Gamma_{\mu\lambda}$ in detail. Instead of the group of transformations (2. 8) we consider firstly only the subgroup

$$\text{a)}\ \ \xi^{0'} = \xi^{0}\,, \qquad \text{b)}\ \ \xi^{\varkappa'} = \xi^{\varkappa'}(\xi^{\varkappa})\,; \quad \text{det.}\left(\frac{\partial\, \xi^{\varkappa'}}{\partial\, \xi^{\varkappa}}\right) \neq 0\,. \quad . \quad (7.\ 1)$$

Then the formulae (2. 6) and (2. 7) simplify and the resulting equations are

$$\Gamma^{\varkappa'}_{\mu'\lambda'} = \Gamma^{\varkappa}_{\mu\lambda} A^{\mu\,\lambda\,\varkappa'}_{\mu'\lambda'\varkappa} + A^{\varkappa'}_{\varkappa} \partial_{\mu'} A^{\varkappa}_{\lambda'}, \quad \ldots \ldots \quad (7.2)$$

$$\Gamma_{\mu'\lambda'} = \Gamma_{\mu\lambda} A^{\mu\,\lambda}_{\mu'\lambda'} \quad \ldots \ldots \ldots \quad (7.3)$$

from which it follows that $\Gamma^{\varkappa}_{\mu\lambda}$ resp. $\Gamma_{\mu\lambda}$ are the components of a connexion and a tensor resp. in the X_n of the $\xi^{\varkappa}$. If $R_{\nu\mu\lambda}^{\cdot\cdot\cdot\varkappa}$ is the curvature affinor of the connexion $\Gamma^{\varkappa}_{\mu\lambda}$ and ∇_λ the symbol of the covariant derivation with respect to $\Gamma^{\varkappa}_{\mu\lambda}$, the conditions of integrability (2. 2) are equivalent to

$$\text{a)}\quad R_{\nu\mu\lambda}^{\cdot\cdot\cdot\varkappa} = 2\,\Gamma_{\lambda[\nu} A^{\varkappa}_{\mu]}, \qquad \text{b)}\quad \nabla_{[\nu} \Gamma_{\mu]\lambda} = 0. \text{ [10]} \quad \ldots \quad (7.4)$$

If we define $R_{\mu\lambda} \overset{\text{def}}{=} R_{\varkappa\mu\lambda}^{\cdot\cdot\cdot\varkappa}$, $V_{\mu\lambda} \overset{\text{def}}{=} R_{\mu\lambda\varkappa}^{\cdot\cdot\cdot\varkappa}$ we find from (7. 4)

$$R_{\mu\lambda} = (1-n)\,\Gamma_{\mu\lambda}, \quad V_{\mu\lambda} = 0. \quad \ldots \ldots \quad (7.5)$$

If now we apply to the unknown variable the transformation (cf. (3. 3))

$$\overset{\varrho}{\xi}{}^0 = \xi^0 - \log \varrho(\xi^{\varkappa}) \quad \ldots \ldots \ldots \quad (7.6)$$

and denote the transforms of $\Gamma^{\varkappa}_{\mu\lambda}$ and $\Gamma_{\mu\lambda}$ by $\overset{\varrho}{\Gamma}{}^{\varkappa}_{\mu\lambda}$ and $\overset{\varrho}{\Gamma}_{\mu\lambda}$ resp., according to (2. 6, 7) and (7. 2, 3) we obtain

$$\overset{\varrho}{\Gamma}{}^{\varkappa}_{\mu\lambda} = \Gamma^{\varkappa}_{\mu\lambda} + 2\,A^{\varkappa}_{(\mu} \partial_{\lambda)} \log \varrho \quad \ldots \ldots \quad (7.7)$$

$$\overset{\varrho}{\Gamma}_{\mu\lambda} = \Gamma_{\mu\lambda} - \Gamma^{\varkappa}_{\mu\lambda}\,\partial_{\varkappa} \log \varrho - (\partial_\mu \log \varrho)(\partial_\lambda \log \varrho) + \partial^2_{\mu\lambda} \log \varrho. \text{ [11]} \quad (7.8)$$

By the elimination of $\partial_\lambda \log \varrho$ from (7. 7) we get the well known projective parameters of THOMAS [12])

$$\Pi^{\varkappa}_{\mu\lambda} \overset{\text{def}}{=} \Gamma^{\varkappa}_{\mu\lambda} - \frac{2}{n+1} A^{\varkappa}_{(\mu} \Gamma^{\nu}_{\lambda)\nu} \text{ [13]} \quad \ldots \ldots \quad (7.9)$$

transforming for the transformations (7. 1*b*) and (7. 6) as follows

$$\Pi^{\varkappa'}_{\mu'\lambda'} = \Pi^{\varkappa}_{\mu\lambda} A^{\mu\,\lambda\,\varkappa'}_{\mu'\lambda'\varkappa} + \frac{2}{n+1} A^{\varkappa'}_{(\mu'} \partial_{\lambda')} \log \Delta + A^{\varkappa'}_{\varkappa} \partial_{\mu'} A^{\varkappa}_{\lambda'};\ \Delta = \det.(A^{\varkappa'}_{\varkappa}), \quad (7.10)$$

that is independently of the parameter ϱ.

If we eliminate $\log \varrho$ from (7. 7) and (7. 8) we obtain the functions

$$\Omega_{\mu\lambda} \overset{\text{def}}{=} \Gamma_{\mu\lambda} + \frac{1}{n+1} \left\{ \Gamma^{\varkappa}_{\mu\lambda} \Gamma^{\nu}_{\varkappa\nu} - \left(\partial_\mu \Gamma^{\nu}_{\lambda\nu} + \frac{1}{n+1} \Gamma^{\nu}_{\mu\nu} \Gamma^{\varrho}_{\lambda\varrho} \right) \right\} \text{ [14]} \quad (7.11)$$

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[10]) Cf. [5], (4. 1), p. 172.

[11]) Cf. [5], (7. 2), p. 174.

[12]) Cf. [7], II, p. 193.

[13]) $\Pi^{\varkappa}_{\mu\lambda}$ correspond to ${}^*\Gamma^t_{rs}$ in [5], (7. 3), p. 174.

[14]) $\Omega_{\mu\lambda}$ correspond to ${}^*b_{rs}$ in [5], (7. 3), p. 174.

which for the transformations (7. 1b) and (7. 6) transform also independently of the parameter ϱ:

$$\left.\begin{aligned} &\Omega_{\mu'\lambda'} = \Omega_{\mu\lambda} A^{\mu\lambda}_{\mu'\lambda'} + \frac{1}{n+1} \Gamma^{\varkappa'}_{\mu'\lambda'} \partial_{\varkappa'} \log \Delta - \\ &-\frac{1}{(n+1)^2} \{ 2 \Gamma^{\varkappa}_{\varkappa\mu} A^{\mu}_{(\mu'} \partial_{\lambda')} \log \Delta + (\partial_{\mu'} \log \Delta)(\partial_{\lambda'} \log \Delta) + \partial^2_{\mu'\lambda'} \log \Delta \}. \text{ [15]} \end{aligned}\right\} \quad (7.12)$$

The functions $\Omega_{\mu\lambda}$, however, may be derived directly from the parameters $\Pi^{\varkappa}_{\mu\lambda}$. In order to find a relation between $\Omega_{\mu\lambda}$ and $\Pi^{\varkappa}_{\mu\lambda}$ let us introduce the following functions

$$\Pi^{\varkappa}_{\nu\mu\lambda} \stackrel{\text{def}}{=} 2\, \partial_{[\nu} \Pi^{\varkappa}_{\mu]\lambda} + 2\, \Pi^{\varkappa}_{[\nu|\varrho|} \Pi^{\varrho}_{\mu]\lambda} \text{ [16]} \quad . \quad . \quad . \quad . \quad (7.13)$$

$$\Phi_{\mu\lambda} \stackrel{\text{def}}{=} \partial_{\mu} \Gamma^{\varkappa}_{\lambda\varkappa} + \frac{1}{n+1} \Gamma^{\nu}_{\mu\nu} \Gamma^{\varkappa}_{\lambda\varkappa} - \Gamma^{\varkappa}_{\mu\lambda} \Gamma^{\nu}_{\varkappa\nu}. \text{ [17]} \quad . \quad . \quad (7.14)$$

Then we obtain from (7. 9), (7. 13) and (7. 14)

$$\Pi^{\varkappa}_{\nu\mu\lambda} = R_{\nu\mu\lambda}^{\cdot\,\cdot\,\cdot\,\varkappa} + \frac{2}{n+1} \{ A^{\varkappa}_{[\nu} \Phi_{\mu]\lambda} - A^{\varkappa}_{\lambda} \partial_{[\nu} \Gamma^{\varrho}_{\mu]\varrho} \} \quad . \quad . \quad (7.15)$$

and from this equation and (7. 5) it follows that

$$\Phi_{\mu\lambda} = \frac{n+1}{n-1} \Pi_{\mu\lambda} + (n+1)\, \Gamma_{\mu\lambda} \quad . \quad . \quad . \quad . \quad . \quad (7.16)$$

where $\Pi_{\mu\lambda} \stackrel{\text{def}}{=} \Pi^{\varkappa}_{\varkappa\mu\lambda}$.

Evidently the relation between the $\Omega_{\mu\lambda}$ and the $\Pi^{\varkappa}_{\mu\lambda}$ is given by

$$\Omega_{\mu\lambda} = \frac{1}{1-n} \Pi_{\mu\lambda}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.17)$$

From (7. 7) it is evident that to every system of the form (2. 1) exists a projective curvature affinor in the X_n of the $\xi^{\varkappa}$ which, however, vanishes because of (7. 4a).

8. In this section the unknown variable ξ^0 will not be transformed and accordingly be denoted by ζ. Then, as we have seen, in the X_n of the $\xi^{\varkappa}$ (obtained from the X_{n+1} of the ξ^{a} by reduction with respect to the invariant congruence (3. 5) [18])) a connexion $\Gamma^{\varkappa}_{\mu\lambda}$ and a tensor $\Gamma_{\mu\lambda}$ are fixed. We deal here only with the case that this tensor has the rank n. In that case we may write $a_{\mu\lambda}$ instead of $\Gamma_{\mu\lambda}$ and use $a_{\mu\lambda}$ as a fundamental tensor in X_n to raise and lower indices. Of course $\Gamma^{\varkappa}_{\mu\lambda}$ is in general not

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[15]) From (7. 10) and (7. 12) it follows that the statement of the last theorem in [5], p. 174, concerning the functions $\Pi^{\varkappa}_{\mu\lambda}$ and $\Omega_{\mu\lambda}$ must be formulated in a somewhat other way.

[16]) Cf. [4], (35. 8), p. 99.

[17]) Cf. [4], p. 100 where A_{ij} stands for $\Phi_{\mu\lambda}$.

[18]) That means: all points lying in the same $\xi^{\varkappa}$ = const. are identified. Cf. [6], p. 45.

equal to the connexion $\{^{\varkappa}_{\mu\lambda}\}$ belonging to this fundamental tensor. According to this fact next to the covariant differential δw_λ with respect to the connexion $\Gamma^{\varkappa}_{\mu\lambda}$ another covariant differential

$$'\delta w_\lambda \overset{\text{def}}{=} a_{\lambda\varkappa}\,\delta a^{\varkappa\nu} w_\nu \quad \ldots \ldots \ldots \quad (8.1)$$

can be defined. The connexion $'\Gamma^{\varkappa}_{\mu\lambda}$ belonging to this second covariant differential can be formed as follows. From (8. 1) it follows that

$$'\delta w_\lambda = d w_\lambda - (\Gamma^{\varkappa}_{\mu\lambda} - Q_{\mu}{}^{\cdot}{}_{\lambda}{}^{\varkappa})\, w_\varkappa\, d\xi^\mu \quad \ldots \ldots \quad (8.2)$$

where

$$Q_{\mu}{}^{\cdot\lambda\varkappa} \overset{\text{def}}{=} \nabla_\mu a^{\lambda\varkappa} \quad \ldots \ldots \ldots \quad (8.3)$$

and accordingly

$$'\Gamma^{\varkappa}_{\mu\lambda} = \Gamma^{\varkappa}_{\mu\lambda} - Q_{\mu}{}^{\cdot}{}_{\lambda}{}^{\varkappa} \quad \ldots \ldots \ldots \quad (8.4)$$

The connexion $'\Gamma^{\varkappa}_{\mu\lambda}$ may be written also in another form using the well known [19]) formula

$$\Gamma^{\varkappa}_{\mu\lambda} = \{^{\varkappa}_{\mu\lambda}\} + \tfrac{1}{2}(Q_{\mu}{}^{\cdot}{}_{\lambda}{}^{\varkappa} + Q_{\lambda}{}^{\cdot\varkappa}{}_{\cdot\mu} - Q^{\varkappa}{}_{\cdot\mu\lambda}) \quad \ldots \ldots \quad (8.5)$$

In consequence of (7. 4b)

$$Q_{[\nu\mu]\lambda} = 0 \quad \ldots \ldots \ldots \quad (8.6)$$

and therefore from (8. 5) it follows that

$$Q_{\mu}{}^{\cdot}{}_{\lambda}{}^{\varkappa} = 2(\Gamma^{\varkappa}_{\mu\lambda} - \{^{\varkappa}_{\mu\lambda}\}) \quad \ldots \ldots \ldots \quad (8.7)$$

with aid of which we obtain from (8. 4)

$$'\Gamma^{\varkappa}_{\mu\lambda} = 2\{^{\varkappa}_{\mu\lambda}\} - \Gamma^{\varkappa}_{\mu\lambda} \quad \ldots \ldots \ldots \quad (8.8)$$

Evidently (8. 8) is equivalent to

$$'\Gamma^{\varkappa}_{\mu\lambda} = a^{\varkappa\varrho}(\partial_\mu a_{\lambda\varrho} - \Gamma^{\nu}_{\varrho\mu} a_{\lambda\nu}) \quad \ldots \ldots \ldots \quad (8.9)$$

If $'\nabla_\lambda$ is the symbol of the covariant derivation with respect to the connexion $'\Gamma^{\varkappa}_{\mu\lambda}$ and if $'R_{\nu\mu\lambda}^{\cdot\cdot\cdot\varkappa}$ is the curvature affinor of this same connexion, it is easily proved that the following equations

$$\text{a)}\ \ 'R_{\nu\mu\lambda}^{\cdot\cdot\cdot\varkappa} = 2\,\Gamma_{\lambda[\nu} A^{\varkappa}_{\mu]}\,, \qquad \text{b)}\ \ '\nabla_{[\nu}\Gamma_{\mu]\lambda} = 0 \quad \ldots \quad (8.10)$$

are satisfied for the connexion $'\Gamma^{\varkappa}_{\mu\lambda}$. Therefore the functions $'\Gamma^{\varkappa}_{\mu\lambda}$ and $\Gamma_{\mu\lambda}$ may be considered as the coefficients of the following system of the partial differential equations of the second order

$$\partial^2_{\mu\lambda}\zeta = {}'\Gamma^{\varkappa}_{\mu\lambda}\,\partial_\varkappa\zeta + \Gamma_{\mu\lambda}\zeta \quad \ldots \ldots \ldots \quad (8.11)$$

in the unknown function ζ and the independent variables $\xi^\varkappa$. The conditions of integrability of this system are (8. 10). The system (8. 11), however, is the well known conjugated system to the system (2. 1) introduced by

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[19]) Cf. [7], I, p. 83.

L. BIANCHI [20]). Hence the system (2. 1) and its conjugated system (8. 11) are simultaneously completely integrable.

We remark that according to (8. 8) the relation of the systems (2. 1) and (8. 11) is mutually reciprocal [21]).

Collecting results we have:

Theorem IX. *If in the system of equations* (3. 2) *the independent variable* $\zeta = \xi^0$ *is fixed, in the* X_n *obtained from the* X_{n+1} *of the* ξ^a, $a = 0, 1, \ldots, n$, *by reduction with respect to the congruence* $\xi^\varkappa = const.$; $\varkappa = 1, \ldots, n$, *a connexion* $\Gamma^\varkappa_{\mu\lambda}$ *and a tensor* $a_{\mu\lambda} = \Gamma_{\mu\lambda}$ *are determined. If* $a_{\mu\lambda}$ *has the rank n this tensor can be used as fundamental tensor and there exists another connexion* $'\Gamma^\varkappa_{\mu\lambda} = 2\left\{{}^{\,\varkappa}_{\mu\lambda}\right\} - \Gamma^\varkappa_{\mu\lambda}$. *If in the equations* (2. 1) $\Gamma^\varkappa_{\mu\lambda}$ *is replaced by* $'\Gamma^\varkappa_{\mu\lambda}$ *a new system of equations of the second order arises, called the conjugated system of* (2. 1) *after* BIANCHI. *The system* (2. 1) *and its conjugated system* (8. 11) *are reciprocal and simultaneously completely integrable.*

Epe [22]) — *Praha,* April 1949.

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20) Cf. [1].

21) Cf. [1], V, p. 320.

22) The substantial part of this paper was written during my studies by Prof. J. A. SCHOUTEN in Epe, Nederland.

23) I read this paper only when my paper had been communicated in Academy. Under the transformation (7. 1) there is the intrinsic connexion $\Gamma^\nu_{\alpha\beta}$ ([8], (2. 2 *b*), p. 206) related with the system (1. 1) which specializes for the system (2. 1) to the connexion (7. 2). The fundamental invariant $P^{\,i}_{j\alpha\beta}$([8], (2. 8 *b*), p. 207) specializes to the tensor (7. 3).