Mathematics. — *On the evaluation of certain integrals occurring in the theory of the freely vibrating circular disk and related problems.* By C. J. BOUWKAMP. (Philips Research Laboratories, Eindhoven, Netherlands.) (Communicated by Prof. H. B. G. CASIMIR.)

(Communicated at the meeting of September 24, 1949.)

In studying the radiation of sound by a freely vibrating, rigid, circular disk for low frequencies, I recently came across the repeated integral

$$I_{n,m}(\varrho) = \int_0^1 P_{2n+1}(\sqrt{1-\varrho'^2})\,\varrho'\,\mathrm{d}\varrho' \frac{1}{2\pi}\int_0^{2\pi} [\varrho^2 - 2\varrho\varrho'\cos\vartheta' + \varrho'^2]^{\frac{1}{2}(m-1)}\,\mathrm{d}\vartheta', \quad (1)$$

in which n and m are non-negative integers, P is the Legendre polynomial, and $0 \leq \varrho \leq 1$. The same integral is encountered in the theory of diffraction of plane waves by circular disks and apertures. The purpose of this paper is to evaluate the integral (1) in terms of known functions and to derive a number of relations that are useful for the physical applications referred to, details of which will be published in Physica.

First of all it will be shown that $I_{n,m}(\varrho)$ can be obtained from $I_{n,m+2}(\varrho)$ by means of differentiation, namely,

$$\mathrm{D}\, I_{n,m+2}(\varrho) = (m+1)^2\, I_{n,m}(\varrho), \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which D is the differential operator

$$\mathrm{D} = \frac{1}{\varrho}\frac{\partial}{\partial\varrho}\left(\varrho\frac{\partial}{\partial\varrho}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Equation (2) even holds if the function $P_{2n+1}(\sqrt{1-\varrho'^2})$ occurring in (1) is replaced by any function $\varphi_n(\varrho')$ for which the corresponding integrals exist. To prove this, let us consider the *wave function* $\psi(z, \varrho)$ defined by

$$\psi(z, \varrho) = \iint_S \varphi(\varrho')\frac{e^{ikr}}{r}\,\mathrm{d}S, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

in which z, ϱ, ϑ are cylindrical coordinates around the axis of the disk S of unit radius. The domain of integration is given by $z' = 0$, $0 \leq \varrho' \leq 1$, $0 \leq \vartheta' \leq 2\pi$, while r denotes the distance between the point of integration $(0, \varrho', \vartheta')$ and the field point $(z, \varrho, 0)$; k is the wave number. This function (4) is annihilated by the operator $\mathrm{D} + k^2 + \partial^2/\partial z^2$, because it is an axially symmetric solution of the wave equation, $\Delta\psi + k^2\psi = 0$. Let us apply this operator to the power-series expansion of ψ with respect to ik, viz.,

$$(\mathrm{D} + k^2 + \partial^2/\partial z^2)\sum_{\nu=0}^{\infty}\{(ik)^\nu/\nu!\}\iint_S \varphi(\varrho')\,r^{\nu-1}\,\mathrm{d}S = 0. \quad . \quad . \quad . \quad (5)$$

Arranging the left-hand member according to powers of ik, we are led to a power series that has to vanish identically. Therefore all coefficients have to vanish; this gives

$$(\mathrm{D} + \partial^2/\partial z^2) \iint_S \varphi(\varrho') r^{-1} \, \mathrm{d}S = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\mathrm{D} \iint_S \varphi(\varrho') \, \mathrm{d}S = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\frac{1}{(\nu-2)!} \iint_S \varphi(\varrho') r^{\nu-3} \, \mathrm{d}S = \frac{1}{\nu!} (\mathrm{D} + \partial^2/\partial z^2) \iint_S \varphi(\varrho') r^{\nu-1} \, \mathrm{d}S. \quad . \quad (8)$$

$$(\nu = 2, 3, \ldots)$$

We now perform the differentiation with respect to z in (8) under the signs of integration, using the identity

$$(\partial^2/\partial z^2) r^{\nu-1} = (\nu-1) r^{\nu-3} + (\nu-1)(\nu-3) z^2 r^{\nu-5}, \quad . \quad . \quad . \quad (9)$$

and we consider the limiting case $z \to 0$ $(0 \leq \varrho \leq 1)$. As is easily seen, the term with z^2 in (9) does not contribute to the limit because the integral $\iint \varphi(\varrho') r^{\nu-5} \, \mathrm{d}S$, in the most unfavourable case $(\nu = 2)$, is only of the order $1/z$. In the limit $z \to 0$ we can replace r by

$$s = \{\varrho^2 - 2\varrho\varrho' \cos \vartheta' + \varrho'^2\}^{\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

so that (8) becomes, for $z \to 0$,

$$\nu(\nu-1) \iint_S \varphi(\varrho') s^{\nu-3} \, \mathrm{d}S = \mathrm{D} \iint_S \varphi(\varrho') s^{\nu-1} \, \mathrm{d}S + (\nu-1) \iint_S \varphi(\varrho') s^{\nu-3} \, \mathrm{d}S,$$

or

$$\mathrm{D} \iint_S \varphi(\varrho') s^{\nu-1} \, \mathrm{d}S = (\nu-1)^2 \iint_S \varphi(\varrho') s^{\nu-3} \, \mathrm{d}S. \quad . \quad . \quad . \quad (11)$$

Equation (2) is included in (11) for $2\pi\varphi(\varrho') = P_{2n+1}(\sqrt{1-\varrho'^2})$ and $\nu = m + 2$. In passing it may be noticed that eq. (7) is trivial, the integral being independent of ϱ. Furthermore, it follows from (6) that

$$\lim_{z \to +0} -(\partial^2/\partial z^2) \iint_S \varphi(\varrho') r^{-1} \, \mathrm{d}S = \mathrm{D} \iint_S \varphi(\varrho') s^{-1} \, \mathrm{d}S. \quad . \quad . \quad . \quad (12)$$

In particular we thus get

$$\lim_{z \to +0} -(1/2\pi)(\partial^2/\partial z^2) \iint_S P_{2n+1}(\sqrt{1-\varrho'^2}) r^{-1} \, \mathrm{d}S = \mathrm{D}\, I_{n,0}(\varrho). \quad . \quad (13)$$

Hence, if we *define* the function $I_{n,-2}(\varrho)$ by the relation

$$I_{n,-2}(\varrho) = \lim_{z \to +0} -(1/2\pi)(\partial^2/\partial z^2) \iint_S P_{2n+1}(\sqrt{1-\varrho'^2}) r^{-1} \, \mathrm{d}S, \quad . \quad . \quad (14)$$

then eq. (2) holds even for $m = -2$. In what follows, however, m will be a non-negative integer unless otherwise stated.

As a first step towards the evaluation of $I_{n,m}(\varrho)$ let us consider this function at $\varrho = 0$. We have

$$I_{n,m}(0) = \int_0^1 P_{2n+1}(\sqrt{1-\varrho'^2})\,\varrho'^m\,\mathrm{d}\varrho' = \int_0^{\pi/2} P_{2n+1}(\cos\vartheta)\sin^m\vartheta\cos\vartheta\,\mathrm{d}\vartheta. \quad (15)$$

Now the Legendre polynomials can be expressed in terms of hypergeometric functions:

$$P_{2n+1}(\cos\vartheta) = (-1)^n \frac{(2n+1)!}{2^{2n}(n!)^2}\cos\vartheta\, F(-n, n+\tfrac{3}{2}; \tfrac{3}{2}; \cos^2\vartheta).$$

If this finite series is substituted in (15) we can integrate term by term by employing well-known expressions for the trigonometric integrals that occur. The result becomes

$$I_{n,m}(0) = (-1)^n \frac{(2n+1)!}{2^{2n+1}(n!)^2}\,\frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2}m+\frac{1}{2})}{\Gamma(\frac{1}{2}m+2)}\,F(-n, n+\tfrac{3}{2}; \tfrac{1}{2}m+2; 1).$$

The remaining hypergeometric series can be summed by Gauss's formula, and after some transformation the final result is found to be

$$I_{n,m}(0) = \tfrac{1}{2}(-1)^n \frac{\Gamma(n+\frac{3}{2})\,\Gamma(\frac{1}{2}m+\frac{1}{2})\,\Gamma(\frac{1}{2}m+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(\frac{1}{2}m+n+2)\,\Gamma(\frac{1}{2}m-n+\frac{1}{2})}. \quad (16)$$

In the second place we shall determine the value of $I_{n,0}(\varrho)$. For that purpose let us consider the *potential function*

$$V = (1/2\pi)\iint_S \varphi\, r^{-1}\,\mathrm{d}S, \qquad \varphi = P_{2n+1}(\sqrt{1-\varrho'^2}). \quad (17)$$

V being a solution of $\triangle V = 0$. This potential function is most simply expressed in terms of oblate-spheroidal potential functions. The corresponding oblate-spheroidal coordinates ξ, η, ϑ adapted to the circular disk (on which $\eta = 0$) can be defined in terms of the cylindrical coordinates by $z = \xi\eta$, $\varrho^2 = (1-\xi^2)(1+\eta^2)$, $(-1 \leq \xi \leq 1)$, $(\eta \geq 0)$, while the angular variable is the same in the two systems. Now the characteristic solutions of the potential equation, for axially symmetric exterior problems, are provided by $P_n(\xi)Q_n(i\eta)$, where Q denotes the Legendre function of the second kind. Moreover, since V is an even function of z, only terms of even degree occur in V. Hence we can write, with as yet unknown constants λ_ν,

$$V = \sum_{\nu=0}^{\infty} i\lambda_\nu P_{2\nu}(\xi)\,Q_{2\nu}(i\eta).$$

The coefficients in this expansion are easy to calculate for any given function φ; recalling the connection between the electric field strength and the charge density in the theory of electricity, we have

$$\varphi(\varrho') = \lim_{z\to+0} -\frac{\partial V}{\partial z} = \lim_{\eta\to 0} -\frac{1}{\xi}\frac{\partial V}{\partial \eta}. \qquad (0 \leq \varrho' \leq 1)$$

On the positive side ($z = +0$) of the disk S one has $\xi = +(1-\varrho^2)^{\frac{1}{2}}$, $\eta = +0$. Consequently, in the case of (17), λ_ν must be determined from

$$\xi P_{2n+1}(\xi) = \sum_{\nu=0}^{\infty} \lambda_\nu Q_{2\nu}'(+i0) P_{2\nu}(\xi) = \sum_{\nu=0}^{\infty} \lambda_\nu P_{2\nu}(\xi)/P_{2\nu}(0).$$

The left-hand member of this equation can be expressed as a linear combination of $P_{2n}(\xi)$ and $P_{2n+2}(\xi)$; thus all λ's are zero except two of them. These exceptions are

$$\lambda_n = \frac{2n+1}{4n+3} P_{2n}(0), \quad \lambda_{n+1} = \frac{2n+2}{4n+3} P_{2n+2}(0).$$

However, we only need the value of V for $z = +0$, $0 \leq \varrho \leq 1$. This value is given by

$$V_0 = \sum_{\nu=0}^{\infty} i \lambda_\nu P_{2\nu}(\xi) Q_{2\nu}(+i0) = \frac{\pi}{2} \sum_{\nu=0}^{\infty} \lambda_\nu P_{2\nu}(0) P_{2\nu}(\xi).$$

This in fact is an explicit expression for $I_{n,0}(\varrho)$; after some transformation we find that

$$\left.\begin{aligned} I_{n,0}(\varrho) = \frac{\Gamma(n+\frac{1}{2})\,\Gamma(n+\frac{3}{2})}{2\,\Gamma(n+1)\,\Gamma(n+2)} \times \\ \times \left[\frac{2n+2}{4n+3} P_{2n}(\sqrt{1-\varrho^2}) + \frac{2n+1}{4n+3} P_{2n+2}(\sqrt{1-\varrho^2})\right]. \end{aligned}\right\} \quad . \quad (18)$$

An alternative expression for $I_{n,0}(\varrho)$ is obtained when the Legendre polynomials in (18) are expressed as hypergeometric functions. In this connection we have

$$P_{2n+1}(\sqrt{1-\varrho^2}) = F(-n, n+\tfrac{1}{2}; 1; \varrho^2), \quad . \quad . \quad . \quad . \quad (19)$$

from which follows

$$\frac{2n+2}{4n+3} P_{2n}(\sqrt{1-\varrho^2}) + \frac{2n+1}{4n+3} P_{2n+2}(\sqrt{1-\varrho^2}) = F(-n-1, n+\tfrac{1}{2}; 1; \varrho^2). \quad (20)$$

If this is substituted in (18) we obtain after some transformation that

$$I_{n,0}(\varrho) = \tfrac{1}{2}(-1)^n \frac{\Gamma(n+\frac{3}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})}{\Gamma(n+1)\,\Gamma(n+2)\,\Gamma(-n+\frac{1}{2})} F(-n-1, n+\tfrac{1}{2}; 1; \varrho^2). \quad (21)$$

It should be noted that eqs (16) and (21) are consistent with each other.

We are also able to evaluate the function $I_{n,-2}(\varrho)$ in virtue of

$$I_{n,-2}(\varrho) = \mathrm{D}\, I_{n,0}(\varrho) = \frac{1}{\varrho}\frac{\partial}{\partial\varrho}\left(\varrho \frac{\partial}{\partial\varrho}\right) I_{n,0}(\varrho). \quad . \quad . \quad . \quad . \quad (22)$$

Two different expressions can be derived, according as (18) or (21) is employed. The first leads to

$$I_{n,-2}(\varrho) = \tfrac{1}{2}(-1)^n \frac{\Gamma(n+\frac{3}{2})\,\Gamma(-\frac{1}{2})\,\Gamma(-\frac{1}{2})}{\Gamma(n+1)\,\Gamma(n+1)\,\Gamma(-n-\frac{1}{2})} \frac{P_{2n+1}(\sqrt{1-\varrho^2})}{\sqrt{1-\varrho^2}}, \quad (23)$$

as is readily verified by using well-known properties of Legendre polynomials. On the other hand eq. (21) yields

$$I_{n,-2}(\varrho) = \tfrac{1}{2}(-1)^n \frac{\Gamma(n+\frac{3}{2})\,\Gamma(-\frac{1}{2})\,\Gamma(-\frac{1}{2})}{\Gamma(n+1)\,\Gamma(n+1)\,\Gamma(-n-\frac{1}{2})} F(-n, n+\tfrac{3}{2}; 1; \varrho^2), \quad (24)$$

which is from (21) most easily obtained with the aid of

$$\mathrm{D}\,F(a, b; 1; \varrho^2) = 4ab\,F(a+1, b+1; 1; \varrho^2). \quad . \quad . \quad . \quad (25)$$

The agreement between (23) and (24) follows from the known identity

$$\frac{P_{2n+1}(\sqrt{1-\varrho^2})}{\sqrt{1-\varrho^2}} = F(-n, n+\tfrac{3}{2}; 1; \varrho^2). \quad . \quad . \quad . \quad . \quad (26)$$

The general expression for $I_{n,m}(\varrho)$ can now be written down almost immediately by simply inspecting (16), (21), and (24). First of all it will be observed that the numerical factors occurring in (21) and (24) are the respective values of $I_{n,m}(0)$, generally given by (16). We thus come to the conjecture that

$$I_{n,m}(\varrho) = I_{n,m}(0)\;F(A, B; 1; \varrho^2),$$

in which A and B depend on n and m. Further, $A = -n-1$ for $m = 0$, and $A = -n$ for $m = -2$. It is thus reasonable to expect that $A = -\frac{1}{2}(2n+m+2)$ for general values of n and m. By a similar argument we expect to have $B = \frac{1}{2}(2n-m+1)$. In other words, our final conjecture is that

$$\left.\begin{aligned} I_{n,m}(\varrho) = \tfrac{1}{2}(-1)^n \frac{\Gamma(n+\frac{3}{2})\,\Gamma(\frac{1}{2}m+\frac{1}{2})\,\Gamma(\frac{1}{2}m+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(\frac{1}{2}m+n+2)\,\Gamma(\frac{1}{2}m-n+\frac{1}{2})} \times \\ \times F(-n-\tfrac{1}{2}m-1, n-\tfrac{1}{2}m+\tfrac{1}{2}; 1; \varrho^2), \end{aligned}\right\} \quad (27)$$

for all non-negative integers n and m. The truth of this conjecture can be proved as follows. (i) Equation (27) is correct for $m = 0$ and $n = 0, 1, 2, \ldots$, as is obvious from (21). (ii) Equation (27) is correct for $m = 1$ and $n = 0, 1, 2, \ldots$: it follows from (1) and (16) that $I_{n,1}(\varrho) = I_{n,1}(0) = \frac{1}{3}\,(n = 0)$ or $0\,(n > 0)$, which values are also given by (27). (iii) The right-hand member of (27) satisfies (2); this is readily verified with the aid of (25). (iv) If the function $I_{n,m}(\varrho)$ is known for some value of m, then $I_{n,m+2}(\varrho)$ can be obtained by integration of the inhomogeneous differential equation (2) under suitable initial conditions. Now the difference of any two solutions of (2), regarded as an equation for $I_{n,m+2}(\varrho)$ while $I_{n,m}(\varrho)$ is given, is at the utmost equal to $y = C_1 \log \varrho + C_2$, this function being the general solution of the corresponding homogeneous differential equation, $\mathrm{D}y = 0$. The first constant of integration, C_1, may be taken equal to zero because all the integrals considered are finite at $\varrho = 0$. The constant C_2 is uniquely determined by the value of $I_{n,m+2}(\varrho)$ at $\varrho = 0$, and this initial value is correctly given by (27). This completes

the proof of eq. (27). As a corollary, it follows from the reasoning under (iv) that

$$I_{n,m+2}(\varrho) - I_{n,m+2}(0) = (m+1)^2 \int_0^{\varrho} u^{-1}\,du \int_0^{u} I_{n,m}(v)\,v\,dv,$$

or, after integration by parts,

$$I_{n,m+2}(\varrho) = I_{n,m+2}(0) - (m+1)^2 \varrho^2 \int_0^1 t\, I_{n,m}(\varrho t) \log t\,dt, \quad . \quad (28)$$

which is the counterpart of eq. (2).

To simplify the discussion of the function $I_{n,m}(\varrho)$ we consider even and odd values of m separately. In the latter case we have

$$\left.\begin{aligned} I_{n,2m+1}(\varrho) = \tfrac{1}{2}(-1)^n \frac{\Gamma(n+\frac{3}{2})\,\Gamma(m+1)\,\Gamma(m+1)}{\Gamma(n+1)\,\Gamma(m+n+\frac{5}{2})\,\Gamma(m-n+1)} \times \\ \times F(-n-m-\tfrac{3}{2}, n-m; 1; \varrho^2). \end{aligned}\right\} \quad (29)$$

This is identically zero if $0 \leq m < n$, because of the last gamma function in the denominator. On the other hand if $m \geq n \geq 0$ the right-hand member is a polynomial in ϱ^2 of degree $m-n$, since in this case the second parameter of the hypergeometric series is the non-positive integer $n-m$. Furthermore, in terms of Jacobi polynomials we have

$$F(-n-m-\tfrac{3}{2}, n-m; 1; \varrho^2) = G_{m-n}(-2m-\tfrac{3}{2}, 1, \varrho^2). \quad . \quad (30)$$

For even values of the second subscript we get

$$\left.\begin{aligned} I_{n,2m}(\varrho) = \tfrac{1}{2}(-1)^n \frac{\Gamma(n+\frac{3}{2})\,\Gamma(m+\frac{1}{2})\,\Gamma(m+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(m+n+2)\,\Gamma(m-n+\frac{1}{2})} \times \\ \times F(-n-m-1, n-m+\tfrac{1}{2}; 1; \varrho^2), \end{aligned}\right\} \quad (31)$$

which is always a polynomial in ϱ^2 of degree $n+m+1$. In terms of Jacobi polynomials one has

$$F(-n-m-1, n-m+\tfrac{1}{2}; 1; \varrho^2) = G_{n+m+1}(-2m-\tfrac{1}{2}, 1, \varrho^2). \quad (32)$$

The first few functions are listed below:

$$I_{0,1}(\varrho) = \frac{1}{3};\ I_{n,1}(\varrho) = 0 \ \ (n > 0);$$

$$I_{0,3}(\varrho) = \frac{2}{15}\left(1 + \frac{5}{2}\varrho^2\right);\ I_{1,3}(\varrho) = -\frac{2}{35};\ I_{n,3}(\varrho) = 0\ (n > 1).$$

$$I_{0,0}(\varrho) = \frac{\pi}{4}\left(1 - \frac{1}{2}\varrho^2\right);\ I_{1,0}(\varrho) = \frac{3\pi}{32}\left(1 - 3\varrho^2 + \frac{15}{8}\varrho^4\right);$$

$$I_{2,0}(\varrho) = \frac{15\pi}{256}\left(1 - \frac{15}{2}\varrho^2 + \frac{105}{8}\varrho^4 - \frac{105}{16}\varrho^6\right);$$

$$I_{3,0}(\varrho) = \frac{175\pi}{4096}\left(1 - 14\varrho^2 + \frac{189}{4}\varrho^4 - \frac{231}{4}\varrho^6 + \frac{3003}{128}\varrho^8\right);$$

$$I_{0,2}(\varrho) = \frac{\pi}{16}\left(1 + \varrho^2 - \frac{1}{8}\varrho^4\right);$$

$$I_{1,2}(\varrho) = -\frac{\pi}{64}\left(1 - \frac{3}{2}\varrho^2 + \frac{9}{8}\varrho^4 - \frac{5}{16}\varrho^6\right);$$

$$I_{2,2}(\varrho) = -\frac{5\pi}{2048}\left(1 - 6\varrho^2 + \frac{45}{4}\varrho^4 - \frac{35}{4}\varrho^6 + \frac{315}{128}\varrho^8\right);$$

$$I_{3,2}(\varrho) = -\frac{7\pi}{2048}\left(1 - \frac{25}{2}\varrho^2 + \frac{175}{4}\varrho^4 - \frac{525}{8}\varrho^6 + \frac{5775}{128}\varrho^8 - \frac{3003}{256}\varrho^{10}\right).$$

$$I_{0,-2}(\varrho) = -\frac{\pi}{2};\ I_{1,-2}(\varrho) = -\frac{9\pi}{8}\left(1 - \frac{5}{2}\varrho^2\right);$$

$$I_{2,-2}(\varrho) = -\frac{225\pi}{128}\left(1 - 7\varrho^2 + \frac{63}{8}\varrho^4\right);$$

$$I_{3,-2}(\varrho) = -\frac{1225\pi}{512}\left(1 - \frac{27}{2}\varrho^2 + \frac{297}{8}\varrho^4 - \frac{429}{16}\varrho^6\right).$$

Though the function $I_{n,m}(\varrho)$ is completely known, the physical problems mentioned at the beginning of the paper require the result in a different form. Equation (23) is remarkable insofar as the Legendre polynomial reappears in the result of integration; that is, the same function as occurred in the integrand of our integrals (1) and (14). Generally, the physical aspect of our problem requires that $I_{n,m}(\varrho)$ be written as a linear combination of the functions

$$F_{2n+1}(\varrho) = \frac{P_{2n+1}(\sqrt{1-\varrho^2})}{\sqrt{1-\varrho^2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Therefore, let

$$I_{n,m}(\varrho) = \sum_{\nu=0}^{\infty} A_\nu(n, m)\, F_{2\nu+1}(\varrho), \quad . \quad . \quad . \quad . \quad . \quad (34)$$

in which, of course, the series on the right actually has a finite number of terms. Using the orthogonality of the Legendre polynomials we have

$$A_\nu(n, m) = (4\nu + 3)\int_0^{\pi/2} P_{2\nu+1}(\cos\vartheta)\, I_{n,m}(\sin\vartheta)\sin\vartheta\cos\vartheta\, d\vartheta =$$

$$= (4\nu + 3)\, I_{n,m}(0)\int_0^{\pi/2} F(-n - \tfrac{1}{2}m - 1, n - \tfrac{1}{2}m + \tfrac{1}{2}; 1; \sin^2\vartheta)$$

$$P_{2\nu+1}(\cos\vartheta)\sin\vartheta\cos\vartheta\, d\vartheta =$$

$$= (4\nu+3) I_{n,m}(0)\sum_{\mu=0}^{\infty}\frac{\Gamma(n+\frac{1}{2}m+2)\,\Gamma(-n+\frac{1}{2}m+\frac{1}{2})}{\Gamma(n+\frac{1}{2}m+2-\mu)\,\Gamma(-n+\frac{1}{2}m+\frac{1}{2}-\mu)\,\Gamma(\mu+1)\,\Gamma(\mu+1)}$$

$$\int_0^{\pi/2} P_{2\nu+1}(\cos\vartheta)\sin^{2\mu+1}\vartheta\cos\vartheta\, d\vartheta =$$

$$= (4\nu + 3)\, I_{n,m}(0)$$

$$\sum_{\mu=\nu}^{\infty}\frac{\Gamma(n+\frac{1}{2}m+2)\,\Gamma(-n+\frac{1}{2}m+\frac{1}{2})}{\Gamma(n+\frac{1}{2}m+2-\mu)\,\Gamma(-n+\frac{1}{2}m+\frac{1}{2}-\mu)\,\Gamma(\mu+1)\,\Gamma(\mu+1)}\, I_{\nu,2\mu+1}(0),$$

in which we have used eqs (15) and (27). Furthermore, on account of (16), we have

$$A_\nu(n,m) = (-1)^\nu (2\nu + \tfrac{3}{2}) I_{n,m}(0)$$

$$\sum_{\sigma=0}^{\infty} \frac{\Gamma(\nu+\frac{3}{2})\,\Gamma(n+\frac{1}{2}m+2)\,\Gamma(-n+\frac{1}{2}m+\frac{1}{2})}{\Gamma(n+\frac{1}{2}m+2-\nu-\sigma)\,\Gamma(-n+\frac{1}{2}m+\frac{1}{2}-\nu-\sigma)\,\Gamma(2\nu+\frac{5}{2}+\sigma)\,\Gamma(\sigma+1)\,\Gamma(\nu+1)} =$$

$$= (-1)^\nu (2\nu + \tfrac{3}{2}) I_{n,m}(0)$$

$$\frac{\Gamma(\nu+\frac{3}{2})\,\Gamma(n+\frac{1}{2}m+2)\,\Gamma(-n+\frac{1}{2}m+\frac{1}{2})}{\Gamma(\nu+1)\,\Gamma(2\nu+\frac{5}{2})\,\Gamma(n+\frac{1}{2}m+2-\nu)\,\Gamma(-n+\frac{1}{2}m+\frac{1}{2}-\nu)}$$

$$F(\nu-n-\tfrac{1}{2}m-1,\ \nu+n-\tfrac{1}{2}m+\tfrac{1}{2};\ 2\nu+\tfrac{5}{2};\ 1).$$

The remaining hypergeometric series can be summed; inserting moreover the known values of $I_{n,m}(0)$ we finally arrive at

$$A_\nu(n,m) = (-1)^{n+\nu}(\nu+\tfrac{3}{4})$$

$$\frac{\Gamma(n+\frac{3}{2})\,\Gamma(\nu+\frac{3}{2})\,\Gamma(m+3)\,\Gamma(\frac{1}{2}m+\frac{1}{2})\,\Gamma(\frac{1}{2}m+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(\nu+1)\,\Gamma(\frac{1}{2}m+2+n-\nu)\,\Gamma(\frac{1}{2}m+\frac{1}{2}-n-\nu)\,\Gamma(\frac{1}{2}m+\frac{7}{2}+n+\nu)\,\Gamma(\frac{1}{2}m+2-n+\nu)}. \tag{35}$$

It is easily verified that the only values of ν for which $A_\nu(n,m)$ differs from zero are those satisfying $0 \leq \nu \leq n - \frac{1}{2}m + \frac{1}{2}$ (m odd) and $0 \leq \max\,(0, n - \frac{1}{2}m - 1) \leq \nu \leq n + \frac{1}{2}m + 1$ (m even).

For the first few values of n and m one has

$$I_{0,-2}(\varrho) = -\frac{\pi}{2} F_1(\varrho); \qquad I_{1,-2}(\varrho) = -\frac{9\pi}{8} F_3(\varrho);$$

$$I_{2,-2}(\varrho) = -\frac{225\pi}{128} F_5(\varrho); \qquad I_{3,-2}(\varrho) = -\frac{1225\pi}{512} F_7(\varrho);$$

$$I_{0,0}(\varrho) = \frac{\pi}{5} F_1(\varrho) + \frac{\pi}{20} F_3(\varrho);$$

$$I_{1,0}(\varrho) = \frac{3\pi}{140} F_1(\varrho) + \frac{\pi}{20} F_3(\varrho) + \frac{5\pi}{224} F_5(\varrho);$$

$$I_{2,0}(\varrho) = \frac{5\pi}{352} F_3(\varrho) + \frac{25\pi}{832} F_5(\varrho) + \frac{525\pi}{36608} F_7(\varrho);$$

$$I_{0,1}(\varrho) = \tfrac{1}{3} F_1(\varrho); \qquad I_{1,1}(\varrho) = I_{2,1}(\varrho) = 0;$$

$$I_{0,2}(\varrho) = \frac{3\pi}{35} F_1(\varrho) - \frac{\pi}{45} F_3(\varrho) - \frac{\pi}{1008} F_5(\varrho);$$

$$I_{1,2}(\varrho) = -\frac{\pi}{105} F_1(\varrho) - \frac{\pi}{220} F_3(\varrho) - \frac{\pi}{728} F_5(\varrho) - \frac{5\pi}{27456} F_7(\varrho);$$

$$I_{0,3}(\varrho) = \frac{4}{15} F_1(\varrho) - \frac{2}{15} F_3(\varrho); \qquad I_{1,3}(\varrho) = -\frac{2}{35} F_1(\varrho);$$

$$I_{0,4}(\varrho) = \frac{2\pi}{21} F_1(\varrho) - \frac{3\pi}{44} F_3(\varrho) + \frac{3\pi}{728} F_5(\varrho) + \frac{\pi}{13728} F_7(\varrho);$$

$$I_{0,5}(\varrho) = \frac{64}{175} F_1(\varrho) - \frac{224}{675} F_3(\varrho) + \frac{8}{189} F_5(\varrho).$$

Eindhoven, July 1949.