Mathematics. — *Existence of Stieltjes integrals.* II. By R. F. DENISTON (Ames, Iowa). (Communicated by Prof. W. VAN DER WOUDE.)

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The Left- and Right-Cauchy–Stieltjes Integral in the Norm Sense

We shall employ the definition of pseudo-additivity of an interval function and the theorem given by GETCHELL [3] which gives necessary and sufficient conditions for the existence of an integral in the norm sense.

For a specified interval function $\overset{(\sim)}{F}(I)$ we say that $\overset{(\sim)}{F}(I)$ is pseudo-additive at a point z if for $x < z < y$

$$\lim_{x \to z,\, y \to z} \text{Lub}\, |\overset{(\sim)}{F}(<x, y>) - \overset{(\sim)}{F}(<x, z>) - \overset{(\sim)}{F}(<z, y>)| = 0.$$

An interval function will be said to be pseudo-additive on an interval $<a, b>$ if it is pseudo-additive at each point of $<a, b>$.

Theorem of Getchell. For a specified interval function, $\overset{(\sim)}{F}(I)$, for the existence of the integral $N^{(\sim)}\int_a^b fdg$ the following conditions are necessary and sufficient:

$$\sigma^{(\sim)} \int_a^b fdg \text{ exist; and } \overset{(\sim)}{F}(I) \text{ be pseudo-additive on } <a, b>.$$

Theorem 2. *The condition of pseudo-additivity for* $\overset{(-)}{F}(I)$ *is equivalent to the following:*

Condition (q): *In each point of* (a, b) *in which the function* g *is discontinuous on the right the function* f *is continuous on the left.*

Proof. A calculation gives

$$\begin{aligned} &|\overset{(-)}{F}(<x, y>) - \overset{(-)}{F}(<x, z>) - \overset{(-)}{F}(<z, y>)| = \\ &= |f(x)\,[g(y) - g(x)] - f(x)\,[g(z) - g(x)] - f(z)\,[g(y) - g(z)]| \\ &= |g(y) - g(z)| \cdot |f(x) - f(z)|. \end{aligned}$$

The condition of pseudo-additivity is hence equivalent to:

Condition (p):

$$\lim_{\substack{x \to z,\, x < z \\ y \to z,\, y > z}} \text{Lub}\, |g(y) - g(z)| \cdot |f(z) - f(x)| = 0.$$

We now show that *condition* (p) *implies condition* (q), by showing that if (q) is not satisfied (p) is not. Let z be a point at which $|g(z+0) - g(z| > 0$, and at which f is not left-continuous. Then there is a sequence of points $\{y_n\}$, $y_n \to z$, $y_n > z$, $n = 1, 2, \ldots$ such that $|g(y_n) - g(z)| > \Delta > 0$, and a sequence of points $\{x_n\}$, $x_n \to z$, $x_n < z$, $n = 1, 2, \ldots$ such that $|f(z) - f(x_n)| > m$. Hence $\lim_{\substack{x \to z, x<z \\ y \to z, y>z}} \text{Lub}\, |g(y) - g(z)| \cdot |f(z) - f(x)| > 0$, and condition (p) is not realized.

We show that *condition* (q) *implies condition* (p). For a point of $(a, b>$ at which g is continuous on the right

$$\lim_{y \to z, y>z} (g(y) - g(z)) = 0;$$

hence condition (p) is satisfied a priori. For a point of $(a, b>$ at which g is discontinuous on the right condition (q) requires

$$\lim_{x \to z, x<z} (f(x) - f(z)) = 0,$$

which gives immediately condition (p).

Theorem 3. *In the case of* f *bounded,* g *of bounded variation in* $<a, b>$ *for the existence of* $N^{(-)} \int_a^b f dg$ *the following conditions are necessary and sufficient:*

(a'). *Same as* (a) *of Theorem* 1.

(b'). *The set of points in* $(a, b>$ *which are left-sided discontinuities of the function* f *is a null set with respect to the left-side continuity function,* g_l, *of* g.

(This theorem has been proved by SCHAERF [8].)

Proof. In accordance with the theorem of Getchell it is sufficient to show that condition (b') is equivalent to the totality of condition (b) of theorem 1 and condition (q) of theorem 2. We consider separately several sets of points which together exhaust $(a, b>$.

For points of $(a, b>$ in which f is continuous on the left conditions (b) and (b') make no assertion, and condition (q) is trivially satisfied.

Let the set L_1 consist of points of $(a, b>$ in which f is discontinuous on the left and g is discontinuous on the right. If both (b) and (q) hold for L_1, by (q) f is left-continuous whenever g is right-discontinuous; and hence L_1 is null. This implies (b'). On the other hand if (b') holds L_1 is a null set with respect to g_l. This requires that L_1 have no points in which g is right-discontinuous. Then this also requires L_1 is null, which gives both (b) and (q) true for the set L_1.

Let the set L_2 consist of points of $(a, b>$ in which f is discontinuous on the left and g is continuous on the right. If z is a point of L_2

$$\Delta g_c(z) = \Delta g_l(z).$$

Hence the g_c-measure and g_l-measure of the point is the same. Then (b) is the same as (b') for L_2. Condition (q) does not concern L_2.

The sets considered exhaust the points of $(a, b>$, and thus the proof is complete.

The Stieltjes Integral in the Pollard–Moore Sense

Theorem 4. *In the case of f bounded, g of bounded variation in $<a, b>$ for the existence of the integral $\sigma\int_a^b f dg$ the following conditions are necessary and sufficient:*

(a''). *On each side of each point of (a, b) and the right side of a and left side of b if g is discontinuous the function f is continuous.*

(b''). *The points of (a, b) in which f is discontinuous is a null set with respect to the continuity function of g.*

Proof. *The conditions are necessary.*

Firstly, it is necessary that both $\sigma^{(-)}\int_a^b f dg$ and $\sigma^{(+)}\int_a^b f dg$ exist. If we let L and R be respectively the set of left and right-sided discontinuities of f and N be the set of discontinuities of f, it is required by (b) of Theorem 1 that L is a null set with respect to g_c in order that $\sigma^{(-)}\int_a^b f dg$ exist. A corresponding condition for the existence of $\sigma^{(+)}\int_a^b f dg$ gives that R is a null set with respect to g_c. Hence N is a null set with respect to g_c, and this is condition (b'').

From (a) of theorem 1 and the corresponding condition for the existence of $\sigma^{(+)}\int_a^b f dg$ it follows immediately that at each "side" mentioned in (a'') f has a sidewise limit. If f is not sidewise continuous suppose that at a point, z, of (e.g.) left discontinuity of g (and hence of h_l) that $f(z-0)$ exists and is different from $f(z)$. Let D_ϵ be a subdivision for ϵ in the sense that whenever $D', D'' \supseteq D_\epsilon$

$$|D' S[f, h_l] - D'' S[f, h_l]| < \epsilon.$$

Such a D_ϵ is guaranteed to exist by the existence of $\sigma\int_a^b f dh_l$. Now let D' have all the points of D_ϵ, the point z, and as its first point to the left of z a point x for which $|f(x) - f(z-0)| < \frac{1}{2}|f(z) - f(z-0)|$, and $|h_l(x) - h_l(z-0)| < \frac{1}{2}|h_l(z) - h_l(z-0)|$. Let the sum $D' S_0[f, h_l]$ contain the same terms as $D\overset{(-)}{S}[f, h_l]$ with the exception of the term $f(x)\cdot[h_l(z) - h_l(x)]$ which may be replaced by $f(z)\cdot[h_l(z) - h_l(x)]$. A calculation gives

$$\begin{aligned} |D'\overset{(-)}{S}[f, h_l] - D' S_0[f, h_l]| &= |f(z) - f(x)|\cdot|h_l(z) - h_l(x)| \\ &= |f(z) - f(z-0) - (f(x) - f(z-0))|\cdot \\ &\quad |h_l(z) - h_l(z-0) - (h_l(x) - h_l(z-0))| \\ &\geqq \tfrac{1}{4}|f(z) - f(z-0)|\cdot|h_l(z) - h_l(z-0)|. \end{aligned}$$

This difference is greater than ϵ for a suitable choice of ϵ. This result contradicts the definition of D_ϵ for such an ϵ. Similar results are obtained if we suppose a point z of right discontinuity of g such that

$$f(z) \neq f(z+0).$$

The conditions are sufficient.

As in the proof of Theorem 1 we make use of the equation

$$\sigma\int_a^b f dh_l + \sigma\int_a^b f dh_r + \sigma\int_a^b f dg_c = \sigma\int_a^b f dg,$$

and show that each integral on the left exists. We again assume without loss of generality that h_l and h_r are non-decreasing.

(*A*). *The case of* $\sigma\int_a^b f dh_l$.

Defining $h_l^{(m)}$ as in the proof of Theorem 1 we choose an m so that

$$2M \sum_{i=m+1}^{\infty} \overset{(-}{\Delta} h_l(x_i) < \epsilon/4.$$

We can find points $\{y_i\}$ $(i = 1, 2, \ldots, m)$ such that y_i is between x_i and the nearest point of the set $\{x_i\}$ lying to the left of x_i and such that by the left-continuity of f at x_i

$$|f(\xi_i) - f(x_i)| < \frac{\epsilon}{4 \sum_{i=1}^{m} \overset{(-)}{\Delta} h_l^{(m)}(x_i)}$$

whenever $y_i \leqq \xi_i \leqq x_i$. Let D_ϵ consist of the $\{x_i\}$ and $\{y_i\}$ together with the points a and b. Only the intervals $\{< y_i, x_i >\}$ contribute to $D_\epsilon S[f, h_l^{(m)}]$, and by the choice of y_i we have for $y_i \leqq \xi_i \leqq x_i$

$$|D_\epsilon S[f, h_l^{(m)}] - \sum_{i=1}^{m} f(x_i) \overset{(-)}{\Delta} h_l^{(m)}(x_i)| < \epsilon/4. \quad . \quad . \quad . \quad . \quad (1)$$

Also by the choice of the y_i, for any two refinements D', D'' of D_ϵ we have

$$|D' S[f, h_l^{(m)} - D_\epsilon S[f, h_l^{(m)}]| < \epsilon/4\,, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$|D'' S[f, h_l^{(m)} - D_\epsilon S[f, h_l^{(m)}]| < \epsilon/4. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

By the choice of m we have

$$|D' S[f, h_l] - D' S[f, h_l^{(m)}]| < \epsilon/4, \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$|D'' S[f, h_l] - D'' S[f, h_l^{(m)}]| < \epsilon/4, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5$$

and also

$$\left| \sum_{i=1}^{m} f(x_i) \overset{(-)}{\Delta} h_l(x_i) - \sum_{i=1}^{\infty} f(x_i) \overset{(-)}{\Delta} h_l(x_i) \right| < \epsilon/4. \quad . \quad . \quad . \quad (6)$$

From (2), (3), (4), and (5) we have

$$|D' S[f, h_l] - D'' S[f, h_l]| < \epsilon$$

which establishes that D_ϵ is a mode of subdivision for ϵ in accordance with the Pollard–Moore limit theory. From (1), (2), (4), and (6) we have the additional result

$$\sigma\int_a^b f dh_l = \sum_{i=1}^{\infty} f(x_i) \overset{(-)}{\Delta} h(x_i)$$

where x_i runs over the points of left discontinuity of h_l (or of g).

(B). The case of $\sigma\int_a^b f dh_r$.

Defining $h_r^{(m)}$ as in the proof of theorem 1 we choose m so that

$$2M \sum_{i=m+1}^{\infty} \overset{(+)}{\Delta} h_r(x_i) < \epsilon/4.$$

In this case we find points $\{y_i\}$, $i = 1, 2, \ldots, m$, such that y_i is between x_i and the nearest point of the set $\{x_i\}$ lying to the right of x_i and such that by the right-continuity of f at x_i

$$|f(\xi_i) - f(x_i)| < \frac{\epsilon}{4 \sum_{i=1}^{m} \overset{(+)}{\Delta} h_r^{(m)}(x_i)}$$

whenever $x_i \leqq \xi_i \leqq y_i$. Letting D_ϵ consist of the $\{x_i\}$ and $\{y_i\}$ together with the points a and b the proof follows almost exactly like (a) above. We have also the additional result:

$$\sigma\int_a^b f dh_r = \sum_{i=1}^{\infty} f(x_i) \overset{(+)}{\Delta} h_r(x_i)$$

where x_i runs over the points of right discontinuity of h_r (or of g).

(C). *The case of* $\sigma\int_a^b f dg_c$.

The conditions are sufficient by a theorem of Bliss (see below) for the existence of the integral in the norm sense or $N\int_a^b f dg_c$. This guarantees existence in the σ-sense.

Remark on the Ordinary Stieltjes Integral.

For the integral $N\int_a^b f dg$ based on $F(<t', t''>) = f(\xi)[g(t'') - g(t')]$ $(t' \leqq \xi \leqq t'')$ GETCHELL gives as the condition of pseudo-additivity the following:

$f(x)$ and $g(x)$ have no common point of discontinuity.

This condition and theorem 4 are seen to be in agreement (in the sense of Getchell's theorem) with the following condition given by BLISS [1] for the ordinary integral:

A necessary and sufficient condition that the norm or Riemann–Stieltjes integral $N\int_a^b f dg$, g of bounded variation, exist is that the total variation of g on the set of the discontinuities of f be zero.

The Modified Integral of Dushnik in the Pollard–Moore Sense

Theorem 5. *In the case of* f *founded,* g *of bounded variation in* $<a, b>$ *for the existence of the integral* $\sigma^{(*)}\int_a^b f dg$ (the Dushnik integral in the Pollard–Moore sense) *the following conditions are necessary and sufficient:*

(a*). *On each side of each point of* (a, b) *and the right side of* a *and left side of* b *if* g *is discontinuous the function* f *has a sidewise limit.*

(b*). *Same as* (b''). (i.e. the points of (a, b) in which f is discontinuous is a null set with respect to the continuity function of g.)

Proof. *The conditions are necessary.*

(a^*) was given by HILDEBRANDT [4] to be necessary and follows from the necessity of sidewise pseudo-additivity of $F^{(*)}$ as given by GETCHELL.

Condition (b^*) *is necessary.*

We suppose without loss of generality that g_c is non-decreasing in $\langle a, b \rangle$. In order to establish a contradiction we suppose that the set N of discontinuities of f in (a, b) has positive outer g_c-measure, but that the integral $\sigma^{(*)}\int_a^b f dg$ exists. Then for arbitrary $\eta > 0$ there is a mode D_η having the property that for all D', D'' satisfying $D', D'' \supseteq D_\eta$ it is true that

$$|D' \overset{(*)}{S}[f, g] - D'' \overset{(*)}{S}[f, g]| < \eta.$$

We denote by N_p the set of points, x, for which there is in every neighborhood a point y such that $|f(x) - f(y)| > p$. Since $N = \sum_{n=1}^{\infty} N_{1/n}$ and since $0 < g_c^* \{N\} \leqq \sum_{n=1}^{\infty} g_c^* \{N_{1/n}\}$ there is a number $p > 0$ for which $g_c^* (N_{1/p}) = m_p > 0$.

Let D_η above be given by

$$a = t_0 < t_1 < \ldots < t_{r_\eta} = b \text{ and consider the set } \sum_{i=1}^{r_\eta - 1} \langle t_i - \partial, t_i + \partial \rangle.$$

By the uniform continuity of g_c on $\langle a, b \rangle$ we are assured of a ∂ such that if x and y satisfy $|x - y| < 2\partial$ we have $|g_c(x) - g_c(y)| < \frac{\epsilon}{2r_\eta}$. Using this ∂ we have that the outer g_c-measure of N'_p, by which we denote $N_p - \sum_{i=1}^{r_\eta - 1} \langle t_i - \partial, t_i + \partial \rangle$, is greater than $m_p - \epsilon/2$. Hence by the meaning of outer measure we can cover the set N'_p by a finite number n_1 of non-overlapping intervals $I_i^{(1)} \equiv \langle x_i, y_i \rangle$, $i = 1, 2, \ldots, n_1$, satisfying

$$\sum_{i=1}^{n_1} g_c(I_i^{(1)}) > \sum_{i=1}^{n_1} g_c^* (I_i^{(1)} \cdot N'_p) > m_p - \epsilon,$$

and having no points in common with the set $\sum_{i=1}^{r_\eta - 1} \langle t_i - \partial, t_i + \partial \rangle$.

Let ∂_1 be the ∂ of uniform continuity for $\epsilon/2n_1$ and let us take points x'_i at a distance ∂_1 to the right of x_i and also points y'_i at a distance ∂_1 to the left of y_i. If any interval had length less than $2\partial_1$ or if $\langle x'_i, y'_i \rangle$ contains no points of N_p we omit it. Let $\{I_i^{(2)}\}$, $i = 1, 2, \ldots, n_2$, be the new (renumbered) set of intervals $\langle x'_i, y'_i \rangle$. We have

$$\sum_{i=1}^{n_2} g_c(I_i^{(2)}) > \sum_{i=1}^{n_2} g_c^* (I_i^{(2)} \cdot N'_p) > m_p - 2\epsilon.$$

The expression on the right is greater than $m_p/2 = m > 0$ if ϵ has been chosen less than $m_p/4$. We now have the following possibilities:

(i). To the right of x'_i there is in $<x'_i, y'_i>$ a first point t_i of N_p, or

(ii). In x'_i or to the right of x'_i in $<x'_i, y'_i>$ there is a first point of those having the property of being a limit point on the right of points of N_p.

If (i) holds take t_i as z_i. If (ii) holds there is in (x'_i, y'_i) a point t_i of N_p. Take this t_i as z_i.

By the nature of the points of N_p, there is near z_i in each interval (x'_i, y'_i) either a point z'_i or a point z''_i satisfying

(iii). $f(z'_i) - f(z_i) > p$, or

(iv). $f(z_i) - f(z''_i) > p$.

Let the modes of subdivision D' and D'' be formed as follows: Both D' and D'' have all points of D_ϵ and the points $\{x_i\}$ and $\{y_i\}$ and differ only in the point taken for ξ_i in forming the sums

$$F^{(*)} = \Sigma f(\xi_i)[g(y_i) - g(x_i)].$$

If (iii) holds take z'_i as ξ_i for D' and z_i as ξ_i for D''. If (iv) holds take z_i as ξ_i for D' and z''_i as ξ_i for D''. Then a calculation gives:

$$D' \overset{(*)}{S}[f, g] - D'' \overset{(*)}{S}[f, g] \geqq p \cdot m,$$

which is greater than η if η has been chosen to be $< p \cdot m$. Since both D' and D'' are refinements of D_η, this result contradicts the definition of D_η.

The conditions are sufficient.

As in the proof of Theorem 1 we make use of the equation

$$\sigma^{(*)}\int_a^b f dh_r + \sigma^{(*)}\int_a^b f dh_l + \sigma^{(*)}\int_a^b f dg_c = \sigma^{(*)}\int_a^b f dg$$

and show that each integral on the left exists. We again assume without loss of generality that h_l and h_r are non-decreasing.

(*A*). *The case of* $\sigma^{(*)}\int_a^b f dh_l$.

Defining $h_l^{(m)}$ as in the proof of theorem 1 we choose an m so that

$$2M \sum_{i=m+1}^{\infty} \overset{(-)}{\Delta} h_l(x_i) < \epsilon/4.$$

We can find points $\{y_i\}$, $i = 1, 2, \ldots, m$, such that y_i is between x_i and the nearest point of the set $\{x_i\}$ lying to the left of x_i and such that by the existence of $f(x_i - 0)$

$$|f(\xi_i) - f(x_i - 0)| < \frac{\epsilon}{4 \sum_{i=1}^{m} \Delta h_l^{(m)}(x_i)}$$

whenever $y_i \leqq \xi_i < x_i$. Let D_ϵ consist of the $\{x_i\}$ and $\{y_i\}$ together with the points a and b. Only the intervals $\{< y_i, x_i >\}$ contribute to $D_\epsilon \overset{(*)}{S}[f, h_l^{(m)}]$ and by the choice of y_i we have for $y_i \leqq \xi_i \leqq x_i$

$$|D_\epsilon \overset{(*)}{S}[f, h_l^{(m)}] - \sum_{i=1}^{m} f(x_i - 0) \overset{(-)}{\Delta} h_l^{(m)}(x_i)| < \epsilon/4. \quad \ldots \quad (1)$$

Also by the choice of the y_i, for any two refinements D', D'' of D_ϵ we have

$$|D' \overset{(*)}{S}[f, h_l^{(m)}] - D_\epsilon \overset{(*)}{S}[f, h_l^{(m)}]| < \epsilon/4, \quad \ldots \ldots \quad (2)$$

$$|D'' \overset{(*)}{S}[f, h_l^{(m)}] - D_\epsilon \overset{(*)}{S}[f, h_l^{(m)}]| < \epsilon/4. \quad \ldots \ldots \quad (3)$$

By the choice of m we have

$$|D' \overset{(*)}{S}[f, h_l] - D' \overset{(*)}{S}[f, h_l^{(m)}]| < \epsilon/4, \quad \ldots \ldots \quad (4)$$

$$|D'' \overset{(*)}{S}[f, h_l] - D'' \overset{(*)}{S}[f, h_l^{(m)}]| < \epsilon/4, \quad \ldots \ldots \quad (5)$$

and also

$$|\sum_{i=1}^{m} f(x_i - 0) \overset{(-)}{\Delta} h_l(x_i) - \sum_{i=1}^{\infty} f(x_i - 0) \overset{(-)}{\Delta} h_l(x_i)| < \epsilon/4. \quad \ldots \quad (6)$$

From (2), (3), (4), and (5) we have

$$|D' \overset{(*)}{S}[f, h_l] - D'' \overset{(*)}{S}[f, h_l]| < \epsilon$$

which establishes that D_ϵ is a mode of subdivision for ϵ in accordance with the Pollard–Moore limit theory. From (1), (2), (4), and (6) we have the additional result

$$\sigma^{(*)} \int_a^b f dh_l = \sum_{i=1}^{\infty} f(x_i - 0) \overset{(-)}{\Delta} h_l(x_i)$$

where x_i runs over the points of left discontinuity of h_l (or of g).

(*B*). *The case of* $\sigma^{(*)} \int_a^b f dh_r$.

Defining $h_r^{(m)}$ as in the proof of theorem 1 we choose m so that

$$2M \sum_{i=m+1}^{\infty} \overset{(+)}{\Delta} h_r^{(m)}(x_i) < \epsilon/4.$$

We find points $\{y_i\}$, $i = 1, 2, \ldots, m$, such that y_i is between x_i and the nearest point of the set $\{x_i\}$ lying to the right of x_i and such that by the existence of $f(x_i + 0)$

$$|f(\xi_i) - f(x_i + 0)| < \frac{\epsilon}{4 \sum_{i=1}^{m} \overset{(+)}{\Delta} h_r^{(m)}(x_i)}$$

whenever $x_i < \xi_i \leqq y_i$. Letting D_ϵ consist of the $\{x_i\}$ and $\{y_i\}$ together with the points a and b the proof follows almost exactly like (*A*) above. We have also the additional result:

$$\sigma^{(*)} \int_a^b f dh_r = \sum_{i=1}^{\infty} f(x_i + 0) \overset{(+)}{\Delta} h_r(x_i)$$

where x_i runs over the points of right discontinuity of h_r (or of g).

(*C*). *The case of* $\sigma^{(*)} \int_a^b f dg_c$.

The conditions are sufficient by a theorem of BLISS (see p. 1124) for the existence of the ordinary integral in the norm sense or $N\int_a^b fdg_c$. This of course is sufficient for the existence of $\sigma^{(*)} \int_a^b fdg_c$.

The Modified Integral of Dushnik in the Norm Sense

Theorem 6. *In the case of f bounded, g of bounded variation in $<a, b>$ for the existence of the integral $N^{(*)}\int_a^b fdg$ the following conditions are necessary and sufficient:*

(a^{**}). In each point of the interval (a, b) and the right side of a and left side of b either f or g is continuous except that both may have removable discontinuity at the same point.

(b^{**}). Same as (b'') and (b^*).

Proof. The following was given by GETCHELL as the condition of pseudo-additivity:

Condition (r). f and g have no points of common discontinuity in (a, b) and the right side of a and left side of b except that both may have removable discontinuity at the same point.

Since (b^{**}) is the same as (b^*) of theorem 5, in order to fulfill the conditions of Getchell's theorem we must show that together (a^*) and (r) are equivalent to (a^{**}).

We consider separately the set N_1 of points of (a, b) and the right side of a and left side of b in which either f or g is continuous and the set N_2 consisting of points for which both functions are discontinuous. For N_1 the three conditions (a^*), (a^{**}), and (r) hold trivially.

For a point of N_2 suppose that (a^*) and (r) both hold. Then (a^{**}) holds since it is the same as condition (r). On the other hand if (a^{**}) holds then (r) holds, being the same. Also the sidewise limits required by (a^*) are guaranteed. Hence (a^*) holds for the set. This completes the proof.

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