MATHEMATICS

A COHOMOLOGY THEORY WITH HIGHER COBOUNDARY OPERATORS. II

VERIFICATION OF THE AXIOMS OF EILENBERG-STEENROD *)

BY

SZE-TSEN HU

(Communicated by Prof. L. E. J. BROUWER at the meeting of November 26, 1949)

1. *Introduction.*

EILENBERG and STEENROD have axiomatised the concepts of homology and cohomology theories (see [1] [1]) and a forthcoming book of them). Under the restricted sense of EILENBERG-STEENROD, a cohomology theory defined on a class of pairs (X, X_0) and continuous maps is a function $H^m(X, X_0)$ defined for every pair (X, X_0) in the class and every integer $m \geqq 0$ with values in the class of abelian groups such that:

(i) If $f: (X, X_0) \to (Y, Y_0)$ is a continuous map in the class, there is an *induced homomorphism*

$$f^*: H^m(Y, Y_0) \to H^m(X, X_0).$$

(ii) For each pair (X, X_0) in the class and $m \geqq 0$, there is a *coboundary homomorphism*

$$\delta: H^m(X_0) \to H^{m+1}(X, X_0).$$

These concepts satisfy the following axioms.

Axiom I. *If f is the identity map on (X, X_0), then f^* is the identity automorphism on $H^m(X, X_0)$.*

Axiom II. *If $f: (X, X_0) \to (Y, Y_0)$ and $g: (Y, Y_0) \to (Z, Z_0)$, then $(gf)^* = f^*g^*$.*

Axiom III. *If $f: (X, X_0) \to (Y, Y_0)$, then $f^*\delta = \delta f_0^*$, where $f_0 = f|X_0$.*

The foregoing axioms are sometimes called the *algebraic axioms.*

Axiom IV. (Homotopy Axiom) *If X is a compact Hausdorff space and X_0 is a closed subset of X and if the continuous maps f, g: $(X, X_0) \to (Y, Y_0)$ are homotopic relative to $\{X_0, Y_0\}$, then $f^* = g^*$.*

Axiom V. (Exactness Axiom) *Given (X, X_0) and the identity maps*

$$i: (X_0, \Phi) \to (X, \Phi), \quad j: (X, \Phi) \to (X, X_0),$$

*) This work was done under Contract N7—ONR—434, Task Order No. III, Navy Department, Office of Naval Research, U.S.A.

1) Numbers in brackets refer to the bibliography at the end of the paper.

the groups and homomorphisms

$$H^0(X, X_0) \xrightarrow{j^*} H^0(X) \xrightarrow{i^*} \dots \xrightarrow{j^*} H^m(X) \xrightarrow{i^*} H^m(X_0) \xrightarrow{\delta} H^{m+1}(X, X_0) \xrightarrow{j^*} H^{m+1}(X) \xrightarrow{i^*} \dots$$

form an exact sequence, [3, p. 687], *called the cohomology sequence of the pair* (X, X_0).

Axiom VI. (Excision Axiom) *If* U *is an open set whose closure is contained in the interior of* X_0, *then the identity map*

$$\mu\colon (X-U,\ X_0-U) \leftarrow (X,\ X_0)$$

induces, for each m, *an isomorphism onto:*

$$\mu^*\colon H^m(X, X_0) \to H^m(X-U, X_0-U).$$

Axiom VII. (Dimension Axiom) *If* X *is a space consisting of a single point* x_0, *then* $H^m(X) = 0$ *for every* $m \neq 0$.

In our note I, [2], a generalisation of the classical cohomology theory is given by introducing coboundary operators of higher order. It is natural to ask if there are analogous axioms which hold good for all of the (p, q)-cohomology groups. The purpose of the present note is to carry out such an investigation.

The algebraic axioms are proved in § 4. The exactness axiom is proved in § 6 and the excision axiom in § 7. For the dimension axiom, the m-dimensional (p, q)-cohomology groups of a single point are computed. This shows that, in general, the dimension axiom does not hold. It remains open whether or not the homotopy axiom is satisfied.

2. *The coboundary homomorphisms.*

Throughout the present note, we assume G to be a discrete abelian group such that the order of every element of G divides $\theta(p, q)$ for a given pair of positive integers p and q, [2, §§ 4 and 5].

Let (X, X_0) be a given pair. We shall define for each integer m a homomorphism

$$\delta^p\colon H^m_{(p,q)}(X_0, G) \to H^{m-p}_{(q,p)}(X \bmod X_0, G),$$

called the *coboundary homomorphism* on the group $H^m_{(p,q)}(X_0, G)$ as what follows.

Let $\iota\colon X_0 \to X$ be the identity map. It follows from a statement of Spanier, [4, p. 411], that the induced cochain transformation, [2, § 7],

$$\iota^{\#}\colon C^m(X, G) \to C^m(X_0, G)$$

is onto. Therefore, for each element $e \in H^m_{(p,q)}(X_0, G)$ there is a cochain $c^m \in C^m(X, G)$ such that $\iota^{\#} c^m$ is a cocycle of order p and represents the element e.

$$\delta^p c^m \in Z^{m+p,q}(X \bmod X_0, G). \tag{2. 1}$$

Proof. Since $\delta^q \delta^p c^m = 0$ by our assumption on the group G, $\delta^p c^m \in Z^{m+p,q}(X, G)$. Since $\iota^{\#} c^m$ is a cocycle of order p, we have

$$\iota^{\#} \delta^p c^m = \delta^p \iota^{\#} c^m = 0.$$

Hence, $\delta^p c^m \in C^{m+p}(X \bmod X_0, G)$. This completes the proof.

(2.2) *$\delta^p c^m$ represents an element $\delta^p e$ of the group $H^{m-p}_{(q,p)}(X \bmod X_0, G)$, which depends only on e.*

Proof. Suppose that $d^m \in C^m(X, G)$ be another cochain such that $\iota^{\#} d^m$ is a cocycle of order p and $\iota^{\#} d^m$ represents the element e. Then it follows that $\iota^{\#} c^m - \iota^{\#} d^m$ is a coboundary of order q, i.e. there exists a cochain $a^{m-q} \in C^{m-q}(X_0, G)$ such that

$$\iota^{\#} c^m - \iota^{\#} d^m = \delta^q a^{m-q}.$$

Since the homomorphism

$$\iota^{\#}\colon C^{m-q}(X, G) \to C^{m-q}(X_0, G)$$

is onto, there is a cochain $b^{m-q} \in C^{m-q}(X, G)$ such that $a^{m-q} = \iota^{\#} b^{m-q}$. Hence we obtain

$$\iota^{\#}(c^m - d^m - \delta^q b^{m-q}) = 0,$$

i.e. the cochain $c^m - d^m - \delta^q b^{m-q}$ belongs to the subgroup $C^m(X \bmod X_0, G)$. Taking its p-th coboundary, we obtain

$$\delta^p c^m - \delta^p d^m \in B^{m+p,p}(X \bmod X_0, G).$$

This completes the proof.

Now, the following theorem is obvious.

(2.3) *The correspondence $e \to \delta^p e$ is a homomorphism*

$$\delta^p\colon H^m_{(p,q)}(X_0, G) \to H^{mp}_{(q,p)}(X \bmod X_0, G).$$

3. *Induced homomorphisms of the groups.*

By a map f of a pair[2] (X, X_0) into a pair (Y, Y_0), we mean a (continuous) map $f\colon X \to Y$ such that $f(X_0) \subset Y_0$. Let f be a given map of (X, X_0) into (Y, Y_0) and denote by $f_0 = f|X_0$ the partial map on X_0 into Y_0. Let

$$\iota_1\colon X_0 \to X, \qquad \iota_2\colon Y_0 \to Y$$

be the identities. Then we have the following diagram of cochain transformations

$$\begin{array}{ccc} C^m(Y, G) & \xrightarrow{\ f^{\#}\ } & C^m(X, G) \\ \downarrow \iota_2^{\#} & & \downarrow \iota_1^{\#} \\ C^m(Y_0, G) & \xrightarrow{\ f_0^{\#}\ } & C^m(X_0, G) \end{array}$$

---

[2]) See [2, § 8].

Since $f\iota_1 = \iota_2 f_0$, the following relation is an direct consequence of (7.4) of our Note I:

(3. 1) $$\iota_1^{\#} f^{\#} = f_0^{\#} \iota_2^{\#}.$$

The following properties can be easily verified:

(3. 2) $$f^{\#}(C^m(Y \bmod Y_0, G)) \subset C^m(X \bmod X_0, G),$$

(3. 3) $$f^{\#}(Z^{m,p}(Y \bmod Y_0, G)) \subset Z^{m,p}(X \bmod X_0, G),$$

(3. 4) $$f^{\#}(B^{m,q}(Y \bmod Y_0, G)) \subset B^{m,q}(X \bmod X_0, G).$$

As a consequence of (3.3) and (3.4), $f^{\#}$ induces a homomorphism

$$f^*\colon H^m_{(p,q)}(Y \bmod Y_0, G) \to H^m_{(p,q)}(X \bmod X_0, G),$$

called the homomorphism induced by the map f. If X_0 and Y_0 are vacuous, then this homomorphism reduces to the following homormophism

$$f^*\colon H^m_{(p,q)}(Y, G) \to H^m_{(p,q)}(X, G)$$

of the absolute groups. Hence the partial map $f_0 = f|X_0$ induces a homomorphism

$$f_0^*\colon H^m_{(p,q)}(Y_0, G) \to H^m_{(p,q)}(X_0, G)$$

of the groups of the subspaces.

4. *The algebraic axioms.*

The following two axioms are immediate consequences of (7.3) and (7.4) of our Note I, [2].

Axiom I. *If $f\colon (X, X_0) \to (X, X_0)$ is the identity map, then the induced homomorphism f^* is the identity automorphism of the group $H^m_{(p,q)}(X \bmod X_0, G)$.*

Axiom II. *If $f\colon (X, X_0) \to (Y, Y_0)$ and $g\colon (Y, Y_0) \to (Z, Z_0)$, then $(gf)^* = f^*g^*$.*

We are going to prove the following axiom:

Axiom III. *If $f\colon (X, X_0) \to (Y, Y_0)$ and $f_0 = f|X_0$, then $f^*\delta^p = \delta^q f_0^*$.*

Proof. Let $e \in H^m_{(p,q)}(Y_0, G)$ be an arbitrary element. Choose a cocycle of order p $a^m \in Z^{m,p}(Y_0, G)$ which represents e. [Let

$$\iota_1\colon X_0 \to X, \qquad \iota_2;\ Y_0 \to Y$$

be the identities. Since $\iota_2^{\#}$ is onto, we may choose a cochain $c^m \in C^m(Y, G)$ such that $\iota_2^{\#} c^m = a^m$. Then the element

$$f^* \delta^p e \in H^{m-p}_{(q,p)}(X \bmod X_0, G)$$

is represented by the cocycle $f^{\#}\delta^p c^m$ of order q. On the other hand, the element $f_0^* e$ is represented by the cocycle $f_0^{\#} a^m$ of order p. Since

$$\iota_1^{\#} f^{\#} c^m = f_0^{\#} \iota_2^{\#} c^m = f_0^{\#} a^m$$

by (3.1), the element $\delta^p f_0^* e$ is represented by $\delta^p f^{\#} c^m$. Since $f^{\#}\delta^p = \delta^p f^{\#}$, our proof is complete.

5. *Mayer cochain complexes.*

A sequence of groups and homomorphisms is termed a *homomorphism sequence* if it can be indexed from the integers so that, if G_r and h_r denote respectively the group and homomorphism with index r, then

$$h_r;\ G_r \rightarrow G_{r+1}, \qquad (r = \ldots, -1, 0, 1, \ldots).$$

A *Mayer cochain complex* K is a homomorphism sequence $K = \{G_r, h_r\}$ such that $h_{r+1}h_r = 0$ for all $r = \ldots, -1, 0, 1 \ldots$, [3, p. 684].

For an arbitrary integer m, we construct a homomorphism sequence

$$K(m, p, q; X, G) = \{G_r, h_r\}$$

by taking

$$G_r = C^{m+[(r+1)/2]p+[r/2]q}(X, G),$$

$$h_r = \begin{cases} \delta^p, & \text{if } r \text{ is even}, \\ \delta^q, & \text{if } r \text{ is odd}, \end{cases}$$

where the symbol $[x]$ denotes the greatest integer not exceeding x. Since $\delta^p\delta^q = 0 = \delta^q\delta^p$, we have

(5.1) *$K(m, p, q; X, G)$ is a Mayer cochain complex.*

To every group G_r of the Mayer cochain complex $K(m, p, q; X, G)$, we associate with the subgroup

$$F_r = C^{m+[(r+1)/2]p+[r/2]q}(X \bmod X_0, G)$$

of G_r, where X_0 is a given subspace of X. It follows from (8.1) of our Note I, [2], that

$$h_r(F_r) \subset F_{r+1}$$

for each integer r. Thus we obtain another Mayer cochain complex

$$K(m, p, q; X \bmod X_0, G) = \{F_r, h_r\}.$$

The following statement is an immediate consequence of the definition of a subcomplex, [3, p. 685].

(5.2) *$K(m, p, q; X \bmod X_0, G)$ is a subcomplex of the Mayer cochain complex $K(m, p, q; X, G)$.*

Let us use the following abridged notations:

$$K^m = K(m, p, q; X, G),$$
$$K^m_* = K(m, p, q; X \bmod X_0, G).$$

Let $H^r(K)$ denote the r-dimensional cohomology group of the Mayer cochain complex K, [3, p. 685]. Then clearly we have

$$H^r(K^m) = \begin{cases} H^{m+rp/2+rq/2}_{(p,q)}(X, G), & \text{(if } r \text{ is even)}, \\ H^{m+(r+1)p/2+(r-1)q/2}_{(q,p)}(X, G), & \text{(if } r \text{ is odd)}; \end{cases}$$

$$H^r(K^m_*) = \begin{cases} H^{m+rp/2+rq/2}_{(p,q)}(X \bmod X_0, G), & \text{(if } r \text{ is even)}, \\ H^{m+(r+1)p/2+(r-1)q/2}_{(q,p)}(X \bmod X_0, G), & \text{(if } r \text{ is odd)}. \end{cases}$$

6. *The exactness axiom.*

Let (X, X_0) be a given pair. Denote by Φ the empty set and consider the identity maps

$$i;\ (X_0, \Phi) \to (X, \Phi), \quad j;\ (X, \Phi) \to (X, X_0).$$

The maps i and j induce homomorphisms:

$$i^*;\ H^m_{(p,q)}(X, G) \to H^m_{(p,q)}(X_0, G),$$
$$j^*:\ H^m_{(p,q)}(X \bmod X_0, G) \to H^m_{(p,q)}(X, G).$$

Further, we have defined in § 2 the coboundary homomorphisms

$$\delta^p;\ H^m_{(p,q)}(X_0, G) \to H^{m-p}_{(q,p)}(X \bmod X_0, G).$$

Thus, for an arbitrary integer m, we obtain a homomorphism sequence:

$$\ldots \xrightarrow{i^*} H^{m-q}_{(q,p)}(X_0, G) \xrightarrow{\delta^q} H^m_{(p,q)}(X \bmod X_0, G) \xrightarrow{j^*} H^m_{(p,q)}(X, G) \xrightarrow{i^*}$$

$$H^m_{(p,q)}(X_0, G) \xrightarrow{\delta^p} H^{m-p}_{(q,p)}(X \bmod X_0, G) \xrightarrow{j^*} H^{m-p}_{(q,p)}(X, G) \xrightarrow{i^*} \ldots$$

called the (m, p, q)-*cohomology sequence* of the pair (X, X_0) over G.

Exactness axiom. *The (m, p, q)-cohomology sequence of a pair (X, X_0) over G is exact,* i.e. *the kernel of each homomorphism is identical with the image of the preceding.*

Proof. Let ι: $X_0 \to X$ be the identity map, then the induced cochain homomorphism

$$\iota^{\#}:\ C^m(X, G) \to C^m(X_0, G)$$

and its kernel is the subgroup $C^m(X \bmod X_0, G)$. Hence, it follows from Noether theorem that $C^m(X_0, G)$ is isomorphic with the factor group

$$C^m(X, G)/C^m(X \bmod X_0, G).$$

Therefore, the Mayer cochain complex

$$K^m_0 = K(m, p, q;\ X_0, G)$$

might be considered as the quotient complex K^m/K^m_*, [3, p. 685]. Then, our axiom follows from a general theorem of KELLEY-PITCHER, [3, p. 688].

7. *The excision axiom.*

Excision axiom. *If U is an open set of X whose closure is contained in the interior of X_0, then the identity map*

$$\mu:\ (X-U,\ X_0-U) \to (X,\ X_0)$$

induces onto isomorphisms

$$\mu^*:\ H^m_{(p,q)}(X \bmod X_0, G) \to H^m_{(p,q)}(X - U \bmod X_0 - U, G)$$

for every integer m.

Proof. According to SPANIER, [4, p. 418], the induced cochain transformation

$$\mu^{\#}: C^m (X \bmod X_0, G) \to C^m (X - U \bmod X_0 - U, G)$$

is an onto isomorphism. Since $\delta^p \mu^{\#} = \mu^{\#} \delta^p$ and $\mu^{\#} \delta^q = \delta^q \mu^{\#}$, our axiom follows immediately. Q.E.D.

8. *The groups for a single point.*

In the present section, we shall be concerned with the topological space X which consists of only a single point x_0. We are going to compute the m-dimensional (p, q)-cohomology group $H^m_{(p,q)} (X, G)$ in terms of the coefficient group G and the integers m, p, q.

Since X consists of a single point x_0, X^{m+1} consists of the point $x_0^{m+1} = (x_0, \ldots, x_0)$ only. Hence

$$C^m (X, G) = \Phi^m (X, G) \approx G.$$

The onto isomorphism is given by associating to each m-function ϕ of $\Phi^m(X, G)$ the element $g = \phi(x_0^{m+1})$ of G.

For an arbitrary m-function $\phi \in \Phi^m(X, G)$, it is easy to see that

$$\delta^p \phi (x_0^{m+p+1}) = \theta (p, m+1) \phi (x_0^{m+1}).$$

Let kG denote the subgroup of G which consists of all elements of the form kg, $g \in G$; and let Gk denote the subgroup of G consisting of all elements $g \in G$ such that $kg = 0$.

The following easily verified identity

$$\theta(p, m+1)\, \theta(q, m-q+1) = \theta(p, q)\, \theta(p+q, m-q-1)$$

shows that $\theta (q, m-q+1) G$ is a subgroup of $G \theta (p, m+1)$. Then, an elementary consideration will give the following isomorphisms onto:

$$A^m_{(p,q)} (X, G) \approx \begin{cases} 0, & (m < 0), \\ G \theta (p, m+1), & (0 \leqq m < q), \\ G \theta (p, m+1)/\theta (q, m-q+1) G & (m \geqq q). \end{cases}$$

Tulane University,
New Orleans, La., U.S.A.

BIBLIOGRAPHY

1. EILENBERG, S. and N. E. STEENROD, Axiomatic approach to homology theory, Proceedings, Nat. Acad. Sci., 31, 117–120 (1945).
2. HU, S. T., A cohomology theory with higher coboundary operators I, Construction of the groups, these Proceedings, 52, 1144–1150 (1949).
3. KELLEY, J. L. and E. PITCHER, Exact homomorphism sequences in homology theory, Annals of Math., 48, 682–709 (1947).
4. SPANIER, E. H., Cohomology theory for general spaces, Annals of Math., 49, pp. 407–427 (1948).