THE FORMATION OF VORTEX SHEETS IN A SIMPLIFIED TYPE OF TURBULENT MOTION

BY

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(*Mededeling no. 64 uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft*)

(Communicated at the meeting of Jan. 28, 1950)

1. *Introduction.*

The observation of a number of features of turbulent fluid motion has led to the conviction that an important part in this motion is played by vortices with axes in the direction of the flow. Such vortices become visible for instance in the so-called "secondary motion" appearing in rectilinear ducts with non-circular cross section [1]). They can also be seen when a heavy wind blows over a sandy beach: the sand taken up by the wind moves in thin layers or ribbons, constantly waving to and fro and folding and curling about the streamlines [2]). In the explanation of relations governing the turbulent friction in the boundary layer along a wall, vortex motion with the axis of rotation parallel to the direction of the main flow is responsible for the difference between the so-called "momentum transfer", proposed by PRANDTL, and the "vorticity transfer", considered by Taylor [3]).

TAYLOR has pointed out that vortices with their axes in the direction of the streamlines can be subjected to extension or contraction in consequence of local acceleration or deceleration of the flow. In the first case the vorticity in the core will increase; in the second case it will decrease. These two processes play a different part in the development of turbulence: the extension is responsible for the appearance of regions of high vorticity

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[1]) Compare e.g. L. PRANDTL, Proceedings IInd Intern. Congress for Applied Mechanics, Zürich 1926, p. 71; or "Führer durch die Strömungslehre" (Braunschweig 1942), p. 133.

[2]) Compare also R. A. BAGNOLD, The Physics of Blown Sand and Desert Dunes (London 1941), p. 176–179. The so-called "spectres aérodynamiques" obtained by RIABOUCHINSKY likewise give the impression that they are due to the effect of vortices with axes along the streamlines. See D. RIABOUCHINSKY, Spectres Aérodynamiques, Bull. Institut Aérodynamique de Koutchino, Fasc. III (Moscou 1909), p. 59 and seqq.; and also Fasc. IV (Moscou 1912), p. 98, where reference is made to such vortices.

[3]) Compare e.g. S. GOLDSTEIN, Modern Developments in Fluid Dynamics (Oxford 1938), vol. I, p. 206–213.

which are the seat of intensive dissipation, while contraction leads to the appearance of regions of low vorticity [4]).

Looking again to the layers of sand mentioned above, the idea comes to the mind that the drawing out of vorticity regions into thin sheets will be a more general process than the drawing out of vortex tubes into thin cores. The idea finds support through the observation of the curling ribbons or veils of smoke which can rise from a lighted cigarette, or of the veils formed by the vapour rising from a cup of hot water or tea. Even flames seem mainly to be formed of thin sheets. Perhaps in this connection one may also point to the luminous veils presented by "network nebulae" such as N.G.C. 6960 and 6992 in the constellation of Cygnus.

Judging by the generality of these phenomena, one may ask if it can be helpful for the understanding of certain features of turbulent flow, if a simplified picture could be developed illustrating the tendency towards the formation of vortex sheets. Since an adequate treatment of the complete hydrodynamic equations has not yet been found, one must be prepared to accept some drastic simplifications in order to arrive at a picture amenable to mathematical treatment. Now if the x-axis is taken in the direction of the main flow, the vorticity component in the x-direction is dependent on the velocity components in the directions of y and z. These same velocity components are responsible for the change of the cross section of vortices. Hence it will be necessary to have the components v and w and the coordinates y and z in the picture. We will, however, leave aside the component u. As the "divergence in the y, z-plane", $\partial v/\partial y + \partial w/\partial z$, in the actual flow is balanced by $\partial u/\partial x$, we are free to assume that this divergence may take any value and we need not introduce an equation of continuity for v and w. We shall then make a drastic simplification by *neglecting pressure gradients in the directions of y and z.* We shall also omit the terms $u(\partial v/\partial x)$, $u(\partial w/\partial x)$ from the equations of motion for v and w. If the frictional terms provisionally are retained, these equations take the form:

$$(1) \qquad \left\{ \begin{aligned} \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= \nu\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \\ \frac{\partial w}{\partial t} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= \nu\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \end{aligned} \right.$$

With respect to the neglected terms the following may be observed: When we suppose that the velocity component u is equal to a constant amount U + a small correction θ, fluctuating about zero more or less periodically when we proceed along a streamline, the main parts

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4) G. I. TAYLOR, Proc. Roy. Society (London) A **164**, 15 (1938). — The case of a vortex tube drawn out into a thin core has also been considered by the present author in Proc. Kon. Ned. Akad. v. Wetensch. Amsterdam, **43**, 11—12 (1940), and in "Advances in Applied Mechanics", vol. I (New York 1948), p. 198.

$U(\partial v/\partial x)$, $U(\partial w/\partial x)$ of the terms $u(\partial v/\partial x)$, $u(\partial w/\partial x)$ can be absorbed into the terms $\partial v/\partial t$, $\partial w/\partial t$ by introducing a moving coordinate system. Further, since the pressure will be mainly determined by $-\varrho U\theta$, we may expect the neglected pressure gradients to be more or less out of phase with v and w. Formally we may introduce all neglected terms as a system of "impressed forces" on the right hand side of the equations. They represent a coupling of the v, w-field with the main motion u, and we might imagine that more or less periodically they will bring about a new initial state, the further history of which can be discussed with the aid of eqs. (1). It is of course not certain that by cutting up the problem in this way we shall arrive at an adequate picture of what happens in actual turbulent fluid motion, and it is possible that relationships appearing in the solution of eqs. (1) may considerably differ from corresponding relations in actual turbulence. However, the solutions of eqs. (1) show certain properties which, even if they may differ in details from what actually happens, nevertheless seem so striking that one is inclined to see in them an indication of effects which must be of importance.

We shall introduce still another simplification: we assume that ν is so small that in regions where the derivatives of v and w are of normal magnitude, the terms with the factor ν can be discarded. We are then left with the system:

$$\left\{\begin{aligned} &\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=0\\ &\frac{\partial w}{\partial t}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=0\end{aligned}\right. \tag{2}$$

2. *Elementary properties of the equations. — Application to a "homogeneous domain".* — The system (2) possesses characteristics determined by:

$$dy/dt = v \quad ; \quad dz/dt = w \tag{3}$$

Along a characteristic we have:

$$dv/dt = 0 \quad ; \quad dw/dt = 0 \tag{4}$$

Hence the characteristics are straight lines.

When the initial state of the field has been given, it is possible, at least in principle, to derive the future state of the field by means of a construction making use of the characteristics. Difficulties, however, arise when characteristics drawn in an y, z, t-space, should meet each other. In order to obtain insight into what may happen, we consider a domain in which v and w are initially given by the linear functions:

$$\left\{\begin{aligned} v &= A\,(y-\bar{y}) + B\,(z-\bar{z})\\ w &= C\,(y-\bar{y}) + D\,(z-\bar{z})\end{aligned}\right. \tag{5}$$

Here $\bar{y}$ and $\bar{z}$ are constants, representing a point which may be either inside or outside the domain. Within this domain the vorticity and the components of the tensor of the rate of deformation have values independent of the coordinates. We shall call such a domain a *"homogeneous domain"*. — It is useful to note that the following treatment remains applicable when the coordinate axes are not rectangular.

In order to find the state of the field at a later instant, we suppose that A, B, C, D are functions of t and substitute expressions (5) into eqs. (2). Since the equations must be satisfied identically in y and z, we obtain:

$$(6)\qquad \begin{cases} \dot{A} + A^2 + BC = 0 \\ \dot{B} + AB + BD = 0 \\ \dot{C} + AC + CD = 0 \\ \dot{D} + BC + D^2 = 0 \end{cases}$$

From these we deduce:

$$\frac{\dot{A} - \dot{D}}{A - D} = \frac{\dot{B}}{B} = \frac{\dot{C}}{C} = -(A + D).$$

Hence if we write: $A - D = q$, we find: $B = c_1 q$, $C = c_2 q$ (where c_1 and c_2 are integration constants) and further $A + D = -\dot{q}/q$. It follows that:

$$A = -\tfrac{1}{2}(\dot{q}/q - q) \quad ; \quad D = -\tfrac{1}{2}(\dot{q}/q + q).$$

There is still one remaining equation of the system (6), for which we take:

$$\dot{A} + \dot{D} + A^2 + D^2 + 2\,BC = 0.$$

Substitution of the expressions found above leads to:

$$\frac{\ddot{q}}{q} - \frac{3\,\dot{q}^2}{2\,q^2} - \frac{1 + 4\,c_1 c_2}{2}\,q^2 = 0.$$

This equation can be integrated by means of standard methods; its integral has the form: $q = 2\,\beta/N$, where:

$$(7)\qquad N = 1 + 2\,\alpha\,t + (\alpha^2 - \gamma)\,t^2 = (1 + \alpha\,t)^2 - \gamma\,t^2$$

α and β being two further integration constants, while $\gamma = \beta^2(1 + 4\,c_1 c_2)$. The full solution of (6) now becomes:

$$(8)\qquad \begin{cases} A = \dfrac{(\alpha+\beta) + (\alpha^2-\gamma)\,t}{N} \quad ; \quad B = \dfrac{2\beta c_1}{N} \\ D = \dfrac{(\alpha-\beta) + (\alpha^2-\gamma)\,t}{N} \quad ; \quad C = \dfrac{2\beta c_2}{N} \end{cases}$$

The following formulae serve to find the constants from the initial values of A, B, C, D:

$$(9)\qquad \begin{cases} 2\,\alpha = A_0 + D_0 & c_1 = B_0/2\,\beta \\ 2\,\beta = A_0 - D_0 & c_2 = C_0/2\,\beta \\ \gamma = \beta^2 + B_0 C_0 \end{cases}$$

We observe that $\alpha^2 - \gamma = A_0 D_0 - B_0 C_0$ and note that α and γ remain invariant when the initial state of the field is described with respect to an arbitrary system of rectilinear coordinates, whether rectangular or not, provided the determinant of the transformation formulae has the value unity.

3. *Discussion of the result.* — A decisive part in the development of the field is played by the circumstance whether N can or cannot become zero for a positive value of t. The following cases must be distinguished.

(I) *N does not become zero for a positive value of t* in the cases:

$$(a) \quad \gamma < 0$$
$$(b) \quad \gamma > 0, \text{ provided } a > 0 \text{ and } \gamma < a^2.$$

In both cases the asymptotic state of the field is given by:

$$A \to t^{-1} \quad ; \quad D \to t^{-1} \quad ; \quad B \text{ and } C \text{ of order } t^{-2}.$$

Hence the asymptotic expressions for the velocity components are:

$$v = (y - \bar{y})/t \quad ; \quad w = (z - \bar{z})/t.$$

This means that in the end we have a symmetric divergent field; all vorticity originally present has become infinitely diluted.

In the transition case:

$$(b^*) \quad a > 0 \quad ; \quad \gamma = a^2$$

we have: $N = 1 + 2\,a\,t$, so that A, B, C, D all become proportional to $1/(1 + 2\,a\,t)$.

(II) *N does become zero for a positive value of t* (to be denoted by t_c) in the cases:

$$(c) \quad \gamma > 0 \quad ; \quad a > 0 \quad ; \quad \gamma > a^2$$
$$(d) \quad \gamma > 0 \quad ; \quad a < 0 \quad ; \quad \gamma > a^2$$
$$(e) \quad \gamma > 0 \quad ; \quad a < 0 \quad ; \quad \gamma < a^2.$$

In all three cases we have:

$$t_c = 1/(\sqrt{\gamma} - a) \tag{10}$$

In case (e) the equation $N = 0$ has a second positive root $t^* = 1/(-\sqrt{\gamma} - a)$, which is greater than t_c.

In case (c) we have $A_0 + D_0 > 0$; the value of $A + D$ changes sign for $t = 2\,a/(\gamma - a^2)$. In cases (d) and (e) the value of $A + D$ is always negative.

If we consider the motion of the particles of the fluid, it is found that a particle which at the instant $t = 0$ had the position y_0, z_0 and velocities $v_0 = A_0\,(y - \bar{y}) + B_0\,(z - \bar{z})$, $w_0 = C_0\,(y - \bar{y}) + D_0\,(z - \bar{z})$, will have come to a point y_c, z_c at the critical instant t_c, such that

$$y_c - \bar{y} = y_0 - \bar{y} + v_0\,t_c \quad ; \quad z_c - \bar{z} = z_0 - \bar{z} + w_0\,t_c.$$

Making use of the value found for t_c we deduce:

$$y_c - \bar{y} = \frac{(\sqrt{\gamma} - \alpha + A_0)(y_0 - \bar{y}) + B_0 (z_0 - \bar{z})}{\sqrt{\gamma} - \alpha}$$

$$z_c - \bar{z} = \frac{C_0 (y_0 - \bar{y}) + (\sqrt{\gamma} - \alpha + D_0)(z_0 - \bar{z})}{\sqrt{\gamma} - \alpha}.$$

It is easily shown that the ratio:

$$\frac{z_c - \bar{z}}{y_c - \bar{y}} = \frac{C_0}{\sqrt{\gamma} + \beta} = \frac{\sqrt{\gamma} - \beta}{B_0} = \frac{2\beta c_2}{\sqrt{\gamma} + \beta}$$

is independent of the values of y_0 *and* z_0. This means that the whole domain in which the velocity components were given by the expressions (5), *has contracted into a single line* passing through $\bar{y}$, $\bar{z}$.

This result can be illustrated by observing that the two straight lines

$$\frac{z - \bar{z}}{y - \bar{y}} = \frac{2\beta c_2}{\sqrt{\gamma} + \beta} \quad ; \quad \frac{z - \bar{z}}{y - \bar{y}} = \frac{2\beta c_2}{-\sqrt{\gamma} + \beta}$$

are the only two straight streamlines of the initial field. They keep their positions for all values of t. When the description of the field is referred to these lines as coordinate axes, we obtain a new set of values of A, B, C, D as follows:

$$A_1 = \frac{\alpha + \sqrt{\gamma} + (\alpha^2 - \gamma)t}{N} \quad ; \quad B_1 = 0$$

$$D_1 = \frac{\alpha - \sqrt{\gamma} + (\alpha^2 - \gamma)t}{N} \quad ; \quad C_1 = 0.$$

It is found that in all three cases (*c*), (*d*), (*e*) the value of D_1 is always negative and becomes negative infinite for $t = t_c$. In the cases (*c*) and (*d*) the value of A_1 is positive, its limit for $t = t_c$ being $(\gamma - \alpha^2)/2\sqrt{\gamma}$. In case (*e*) A_1 is negative for $t < t_c$; the limiting value is given by the same expression. Hence in cases (*c*) and (*d*) we have convergence towards the first axis, leading to complete contraction of the field into this axis, whilst there is divergence away from the second axis. In case (*e*) there is convergence towards both axes, but the convergence towards the first axis operates more rapidly and again leads to complete contraction of the field into this axis at the instant t_c.

4. *Application to a patchwork of homogeneous domains.* — The application of the method of characteristics to the integration of eqs. (2) in the case of arbitrary initial conditions, is beset with difficulties. However, since we are engaged in the construction of a model, we suppose that the initial state of the field can be represented with sufficient accuracy by a patchwork of homogeneous domains. Applying the analysis of sections 2 and 3 to each separate homogeneous domain, we may find that after a certain lapse of time one of these domains will have contracted into a segment

of a straight line. This line segment will then represent a discontinuity for both components of the velocity; hence it is a line of sinks and at the same time a vortex line. We may consider it as the cross section of a vortex sheet, which is being extended in the direction perpendicular to the y, z-plane.

The length of the segment and the values of the velocities on both sides will depend on the original extent of the domain which has contracted into it. By way of example we suppose the initial distribution of v and w to be as indicated in fig. 1. In the interior region, bounded by the square $|y| + |z| = 1$, we have:

$$v_0 = -2y \quad ; \quad w_0 = y + z,$$

so that here: $A_0 = -2$; $B_0 = 0$; $C_0 = 1$; $D_0 = 1$, and $\alpha = -\frac{1}{2}$; $\beta = -\frac{3}{2}$; $\gamma = \frac{9}{4} > \alpha^2$. The original field thus shows convergence and vorticity. We find $t_c = 0{,}5$; the field contracts into a segment of the line $y = 0$.

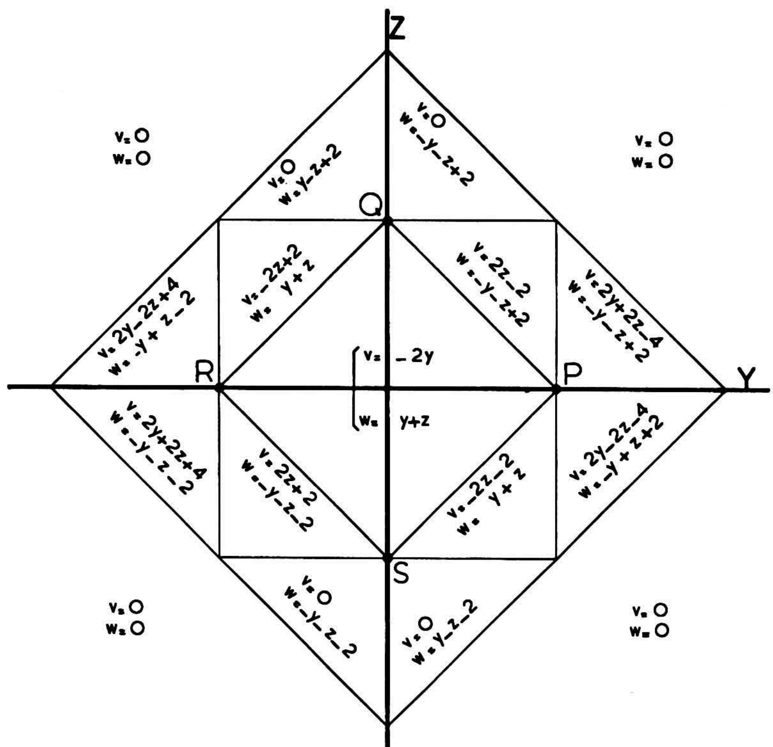

Fig. 1.

In the surrounding network of triangular areas the expressions for v and w have been chosen so as to make both components zero on the four sides of the square $|y| + |z| = 2$ and outside, while care has been taken that there are no discontinuities of v and w in the initial field (there are only discontinuities in their derivatives). With the aid of the formulae of sections 2 and 3 it is found that there is convergence in the two upper triangles and in the two lower ones; however, since $t_c = 1$ for these domains, no difficulty will occur at the instant $t = 0{,}5$. For the other domains t_c is infinite or imaginary.

Some features of the state of the field at the instant $t = 0{,}5$ have been indicated in fig. 2. The outer square, which has retained its position, is

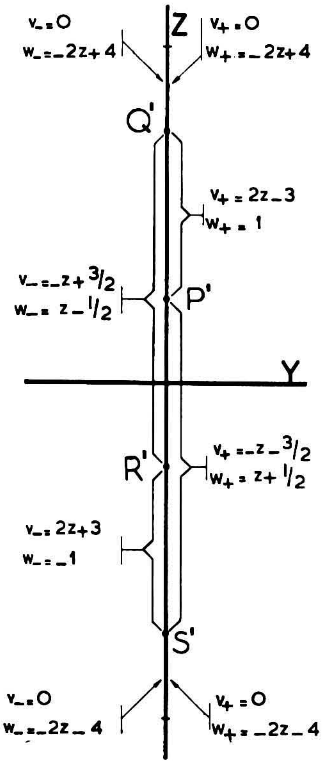

Fig. 2.

not pictured; the inner square has contracted into the segment of the z-axis from $z = +\,1{,}5$ to $z = -\,1{,}5$. The values of v and w on both sides of this segment have been inscribed in the diagram. — It is of interest to note the values of the following quantities:

segment	:	$Q'P'$	$P'R'$	$R'S'$
$c = \frac{1}{2}(v_- + v_+)$	:	$\frac{1}{2}z - \frac{3}{4}$	$-z$	$\frac{1}{2}z + \frac{3}{4}$
$\Delta = \frac{1}{2}(v_- - v_+)$	:	$-\frac{3}{2}z + \frac{9}{4}$	$+\frac{3}{2}$	$+\frac{3}{2}z + \frac{9}{4}$
$\Gamma = \; w_+ - w_-$	:	$-z + \frac{3}{2}$	1	$+z + \frac{3}{2}$

The quantity 2Δ, which measures the convergence into the segment, is essentially positive. The quantity Γ measures the strength of the vortex sheet.

5. *Structure and motion of the region of convergence.* — The "line (or segment) of convergence", obtained in the analysis of the preceding section in reality must be considered as an approximation to a narrow region. For a more accurate picture we make use of the equations (1) in which the terms with ν have been retained.

We consider a small region surrounding a short segment of a "line of convergence", which we suppose to have a radius of curvature different from zero. We limit ourselves to a short interval of time. At the instant of observation we introduce a system of coordinates with the origin in the centre of the segment, the z-axis along the tangent and the y-axis in the direction of the normal. We can assume that derivatives with respect to y will be much larger than derivatives with respect to z in the region of convergence: the former will be of the order ν^{-1}, while the latter will be of normal magnitude. Since the region may be in motion, derivatives with respect to t can become large. However, for a short while we can introduce a moving system of auxiliary coordinates by means of the formulae:

$$y' = y - f(t; z) \quad ; \quad z' = z \quad ; \quad t' = t,$$

where we assume $\partial f/\partial z$ to be zero at the instant considered ($\partial^2 f/\partial z \partial t$ may be different from zero). We write $\partial f/\partial t = c$ and obtain:

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'} \quad ; \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \quad ; \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - c\frac{\partial}{\partial y'}.$$

When these expressions are substituted into eqs. (1) it is possible to determine c in such a way that $\partial v/\partial t'$ becomes of normal order of magnitude at the instant considered. We assume that the same value of c also makes $\partial w/\partial t'$ to be of normal order of magnitude, which assumption is allowed provided the region of convergence displaces itself without a rapid change of internal structure. The equations will contain terms of various orders, the highest being of order ν^{-1}. If only these terms are retained, the equations reduce to:

$$\text{(11)} \qquad \left\{ \begin{aligned} -c\frac{\partial v}{\partial y'} + v\frac{\partial v}{\partial y'} &= \nu\frac{\partial^2 v}{\partial y'^2} \\ -c\frac{\partial w}{\partial y'} + v\frac{\partial w}{\partial y'} &= \nu\frac{\partial^2 w}{\partial y'^2} \end{aligned} \right.$$

The first equation has the integral:

$$\tfrac{1}{2}(v - c)^2 - \nu(\partial v/\partial y') = \text{function of } z, \text{ say } g(z).$$

This expression must be valid through the region of rapid change of v and on both sides of it, where the derivative $\partial v/\partial y'$ takes values of normal magnitude, so that $\nu(\partial v/\partial y')$ can be neglected. Again using subscripts $_-$ and $_+$ to distinguish between values on the two sides of the region, we must have:

$$\tfrac{1}{2}(v_- - c)^2 = \tfrac{1}{2}(v_+ - c)^2 = g(z),$$

from which:

$$\text{(12)} \qquad c = \tfrac{1}{2}(v_- + v_+)$$

The quantity c determines the velocity of displacement of the region of convergence in the direction normal to its longitudinal extension. Since it can be a function of z (in the cases considered it is a linear function of z over a certain interval, passing then into another linear function, etc.), segments of the line of convergence will usually show a rotation simultaneously with their displacement. In general the line may deform in a complicated way.

The final expression for v becomes:

$$v = \tfrac{1}{2}(v_- + v_+) - \triangle \cdot \tanh(y'\triangle/2\nu) \tag{13a}$$

and the coresponding expression for w:

$$w = \tfrac{1}{2}(w_- + w_+) + \tfrac{1}{2}\Gamma \cdot \tanh(y'\triangle/2\nu) \tag{13b}$$

The integrals of $\nu\,(\partial v/\partial y)^2$ and $\nu\,(\partial w/\partial y)^2$ taken over the thickness of the region of convergence, have values $\frac{2}{3}\triangle^3$, $\frac{1}{6}\Gamma^2\triangle$ respectively; these values are independent of ν.

The results (13a) and (13b) lead to transition regions for v and w of a thickness of order ν. This is a consequence of our neglect of the effects of pressure gradients in the y, z-plane, which made it possible that fluid disappearing from the field is carried away in an extremely narrow region. In this respect our picture will deviate from the actual behaviour of a fluid. A picture more approaching actual conditions may perhaps be given by an example in which $u = U + A\,x$; $v = -A\,y$; $w =$ function of y and t; $p = -\frac{1}{2}\varrho\{(U + A\,x)^2 + (A\,y)^2\}$, U and A being constants. The equations for u and v are satisfied exactly. The equation for w has the form:

$$\frac{\partial w}{\partial t} + v\frac{\partial w}{\partial y} = \nu\frac{\partial^2 w}{\partial y^2},$$

which outwardly is of the same type as before. The circumstance, however, that now v decreases linearly with y when we go towards $y = 0$, so that the "region of convergence" is of finite width, has the effect that the vorticity is much less concentrated than in the preceding case. Solutions derived from the above equation for various initial conditions indicate the appearance of a vortex region with thickness of order $\sqrt{\nu}$. The integral of $\nu\,(\partial w/\partial y)^2$ over the thickness of this region then becomes of the order $\sqrt{\nu}$. This point, however, requires further investigation.

Since the main result of this section is the expression (12) for c, we may leave aside the problem of the thickness of the vortex sheet and be content with the law for its displacement.

One may also assume a motion of this sheet in the tangential direction, determined by the value of $\frac{1}{2}(w_- + w_+)$. This is of importance mainly at the endpoints of its section with the y, z-plane. In between the endpoints the value of the normal velocity c is the more important quantity.

It will be evident that the displacement of the vortex sheet, taken into consideration together with the change of the domains of the field adjacent to it, as determined by eqs. (2), usually will entail a gradual change of the values of v_-, v_+, w_-, w_+, and thus also of c and of the vortex strength Γ.

6. *Statistical problems.* — We have arrived at a result which in principle enables us to follow the development of any field, the initial state of which is formed by a patchwork of "homogeneous domains" with rectilinear boundaries between them. We must distinguish between divergent domains and domains which are contracting; for those of the second class we must find out which one has the smallest value for t_c. Having determined the position of the segment into which this domain contracts, we must obtain the motion of this segment after the instant t_c from the fields of flow adjacent to it. Simultaneously we must investigate which domain is the next one to contract into a segment. From the instant when this has occurred, we must follow the motions of both segments.

Gradually the number of these segments will increase, until practically all homogeneous domains belonging to class II will have disappeared. The only domains left will be those of class I. As shown in section 3 all of these asymptotically approach to simple symmetric fields with $v = (y - \bar{y}_i)/t$, $w = (z - \bar{z}_i)/t$, each domain having its own values of $\bar{y}_i$, $\bar{z}_i$. These domains will be separated by a polygonal network of line segments, which are lines of sinks and of concentrated vorticity. It is probable that certain conditions must be satisfied at these segments which still must be investigated.

The lines of sinks and of concentrated vorticity (vortex sheets) are the objects of main interest in the field, and it is their motion which attracts attention. Neighbouring lines may approach to each other and finally may meet. In such a case the domain between them is gradually squeezed out of the field; the two line segments coalesce and from then onward form a single entity. Hence after an initial period of formation of lines of sinks and of concentrated vorticity, there comes a period in which the number of these lines is gradually reduced. This will entail a gradual increase of the average cross dimensions of the remaining homogeneous domains with divergent motion.

At the same time there will be dissipation of energy in the vortex sheets, which will be related to a gradual decrease of the energy content of the whole field. It is to be expected that the thickness of the regions of convergence or of the vortex sheets — which determines the limit to the finer structure of the field — will gradually increase likewise, although the rate may be different from the rate of increase of the coarse scale determined by the cross dimensions of the homogeneous domains.

The picture we have developed could be taken as a basis for a statistical treatment of the particular type of turbulence we have considered, in which the vorticity is concentrated into vortex sheets. We could start from an array of initial states, in order to find out which are the common characteristics of their development in course of time. The picture would refer to the case of decaying (or "free") turbulence, the initial state being continually produced anew by some outward agency, as is the case with the turbulence of the air stream in a wind channel, passing through a screen. The main interest of the picture will be the motion and the gradual coalescence of the vortex sheets, and a means for attacking the problem would be to study the correlation between velocity components in different points of the field. This correlation is influenced by the chance for two points at a given distance from each other to belong or not to belong to the same homogeneous domain.

The construction of a statistical theory for a two-dimensional field of this type will require the consideration of many parameters, even when we restrict to the asymptotic state.

However, now we have arrived at the picture of vorticity concentrated into particular lines or sheets, moving and gradually coalescing, we may go a step further in our simplification and introduce a *one-dimensional picture*. We then consider a single variable v, which is a function of the time and of only one coordinate y, and is subjected to the equation:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = \nu \frac{\partial^2 v}{\partial y^2} \tag{14}$$

Again regions in which $\partial v/\partial y < 0$ will behave differently from regions in which $\partial v/\partial y > 0$. The former contract into narrow domains and lead to the appearance of jumps in the curve of v. Such jumps will be of importance when we study mean values of the type $\overline{v(y)\, v(y+\eta)}$ with a fixed value of η. The equation moreover has the advantage that the corresponding relation for the energy of the field is simpler than it is for eqs. (1).

Properties of the solutions of the equation mentioned have been investigated in some previous papers. With regard to physical dimensions, it conforms to the type of the hydrodynamical equations; hence one may expect that similarity considerations of related character will be applicable. In a future communication it is intended to develop some statistical formulae referring to the asymptotic form of these solutions.