MATHEMATICS

SOME METRICAL THEOREMS OF DIOPHANTINE APPROXIMATION. IV

BY

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1. *Introduction.*

This paper is in some ways a continuation of III and uses the same notation as far as possible. It is, however, completely self-contained.

Throughout this paper let

$$f_1(\theta),\ f_2(\theta), \dots,\ f_n(\theta), \dots$$

be a sequence of functions with positive monotonely non-decreasing continuous derivatives all defined in the range [1])

$$0 \leqslant \theta \leqslant 1.$$

We shall denote by $\{x\}$ the fractional part of x. For $0 \leqslant \alpha \leqslant \beta \leqslant 1$ we define $F_N(\alpha, \beta : \theta)$ to be the number of $n \leqslant N$ such that

$$(1) \qquad \alpha \leqslant \{f_n(\theta)\} < \beta.$$

Further we put

$$R_N(\alpha, \beta : \theta) = F_N(\alpha, \beta : \theta) - N(\beta - \alpha)$$

so that $R_N(\alpha, \beta : \theta)$ is the excess of the number of solutions of (1) over the number to be expected at random.

Finally we put

$$\mathfrak{R}_N(\theta) = \operatorname*{Max}_{\alpha, \beta} |R_N(\alpha, \beta : \theta)|$$

so that $\mathfrak{R}_N(\theta) = ND(N)$ in KOKSMA's notation.

In III of this series I proved, and ERDÖS-KOKSMA [1] proved it independently and practically simultaneously by a different method, that if $f'_m(\theta) - f'_n(\theta)$ is monotonic for all m, n and if

$$(2) \qquad |f'_m(\theta) - f'_n(\theta)| \geqslant K > 0$$

for some K and all m, $n \neq m$, θ then for almost all θ

$$(3) \qquad \mathfrak{R}_N(\theta) = O(N^{1/2} \log^{(5/2)+\varepsilon} N).$$

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[1]) The extension of the results proved to a more general interval $a \leqslant \theta \leqslant b$ instead of $0 \leqslant \theta \leqslant 1$ is trivial.

In this paper I show that the estimate (3) can be improved if the derivatives $f'_n(\theta)$ increase fast enough.

We denote by A_0, A_1, A_2, ... positive absolute constants and by A an absolute positive constant, not necessarily the same in all contexts.

Theorem. *Suppose there is a positive function $\varphi(n)$ of the positive integer variable n such that*

$$\text{(i)} \quad f'_n(\theta) \geqslant e^{\varphi(n)} f'_{n-1}(\theta) \ , \ f'_1(\theta) \geqslant 1$$

for all θ and $n > 1$.

$$\text{(ii)} \quad \log n \log \log n \geqslant \varphi(n) \geqslant c > 0$$

when n is large enough, where c is some constant independent of n.

$$\text{(iii)} \quad \varphi(n) \text{ and } \log n \log \log n \, \varphi^{-1}(n)$$

are monotonic non-decreasing when n is large enough.

Then for almost all θ there is an $N_0 = N_0(\theta)$ such that

$$\text{(4)} \qquad \mathfrak{R}_N(\theta) \leqslant A_0 N^{1/2} \log^{1/2} N \log \log N \, \varphi^{-1/2}(N) \qquad (N > N_0)$$

where A_0 is some absolute constant.

We note the two extreme cases.

$\varphi(n) = \log n \log \log n$. Then (4) becomes $\mathfrak{R}_N(\theta) \leqslant A_0 N^{1/2} \log \log^{1/2} N$. This is best possible apart from the constant A_0. Indeed it is an immediate consequence of the statistical "law of the iterated logarithm" when μ_n is any strictly increasing sequence of integers and $f_n(\theta) = 2^{\mu_n}\theta$ that

$$\lim_N . \sup . \frac{\mathfrak{R}_N(\theta)}{N^{1/2} \log \log^{1/2} N} \geqslant \lim_N . \sup . \frac{R_N(0, 1/2 : \theta)}{N^{1/2} \log \log^{1/2} N} = 2^{-1/2}$$

for almost all θ, however quickly μ_n tends to infinity. [cf. KHINTCHINE [3]].

$\varphi(n) = c = \text{constant}$. This is the "lacunary" case and then (4) becomes $\mathfrak{R}_N(\theta) \leqslant A_0 c^{-1/2} N^{1/2} \log^{1/2} N \log \log N$. This estimate is stronger than one obtained by ERDÖS-KOKSMA [2], who, moreover, required a condition on $f''_n(\theta)$.

2. *Notation.*

We use the following symbols:—

$[x]$: the greatest integer not greater than x.

$||x||$: the difference between x and the nearest integer, i.e.
$$||x|| = \min_{n=0, \pm 1, \pm 2, \ldots} |n - x|.$$

$\{x\}$: the fractional part of x. Thus $x = [x] + \{x\}$.

Define a function $r(\alpha, \beta : x)$ of the variables α, β, x by

$$\begin{aligned} r(\alpha, \beta : x) &= 1 - \{\beta - \alpha\} \text{ if } \{x - \alpha\} < \{\beta - \alpha\} \\ &= - \{\beta - \alpha\} \text{ otherwise.} \end{aligned}$$

It is easily verified that

$$(5) \qquad R_N(\alpha, \beta : \theta) = \sum_{n \leqslant N} r(\alpha, \beta : f_n(\theta))$$

when $0 \leqslant \alpha \leqslant \beta \leqslant 1$ (i.e. whenever the symbol $R_N(\alpha, \beta : \theta)$ has been defined). We shall use the equation (5) to define $R_N(\alpha, \beta : \theta)$ for other values [1]) of α, β. More generally we define

$$_{N_1}R_{N_2}(\alpha, \beta : \theta) = \sum_{N_1 < n \leqslant N_2} r(\alpha, \beta : f_n(\theta)).$$

We note also

Lemma 1. *For each integer* $n \geqslant 1$ *the function* $r = r(\alpha, \beta : x)$ *satisfies*

$$r^n = U_n r + V_n \gamma$$

where $\gamma = ||\beta - \alpha||$ *and* U_n, V_n *are independent of* x. *Further*

$$|U_n| \leqslant 1, \quad |V_n| \leqslant 1, \quad V_1 = 0.$$

For $r(\alpha, \beta : x)$, when α, β are fixed, takes only the values $1 - \gamma$ and $-\gamma$. The identity then follows with

$$U_n = (1 - \gamma)^n - (-\gamma)^n$$
$$V_n = (1 - \gamma)\, U_{n-1}$$

and the rest is trivial.

3. *An estimation lemma.*

Our proof depends essentially on the following lemma:—

Lemma 2. *Suppose* $g_1, \ldots, g_l$ *are any* l *of the functions* $f_1, f_2, \ldots$ *(*l *any integer). Write* $r(x)$ *for* $r(\alpha, \beta : x)$ *where* $\gamma = ||\beta - \alpha||$ *and* α, β *are any numbers. Then*

$$\left| \int_0^1 r(g_1(\theta))\, r(g_2(\theta)) \ldots r(g_l(\theta))\, d\theta \right|$$

$$\leqslant 2\gamma \left\{ \frac{2l-1}{g_l'(0)} + 2 \int_0^1 \frac{g_1'(\theta) + \ldots + g_{l-1}'(\theta)}{g_l'(\theta)}\, d\theta \right\}.$$

The case $l = 2$ is practically the special case $h_m(x) = h_n(x) = r(x)$, $\varphi_m = \varphi_n = \gamma$ of lemma 2 in paper II. The proof for general l runs similarly.

Write

$$r^*(x) = \int_0^x r(\xi)\, d\xi$$

so that

$$r^*(1) = \int_0^1 r(x)\, dx = 0, \qquad |r^*(x)| \leqslant \gamma\,.$$

[1]) This extension of the meaning of $R_N(\alpha, \beta : \theta)$ does not affect the definition of $\mathfrak{R}_N(\theta)$ as

$$R_N(\alpha, \beta : \theta) = R_N(\{\alpha\}, \{\beta\} : \theta) = -R_N(\{\beta\}, \{\alpha\} : \theta)$$

and at least one of the last two expressions has a meaning in the original sense.

We first show that if $g(\theta)$ is one of the $g_j(\theta)$

$$(6) \qquad \left|\int_a^b r(g(\theta))\,d\theta\right| < \frac{2\gamma}{g'(a)}$$

whenever $0 \leqslant a \leqslant b \leqslant 1$. Indeed

$$\int_a^b r\,(g(\theta))\,d\theta = \int_{\theta=a}^{b} \frac{r(g(\theta))}{g'(\theta)}\,d\,(g(\theta)) = \left[\frac{r^*(g(\theta))}{g'(\theta)}\right]_{\theta=a}^{b} - \int_{\theta=a}^{b} r\,(g\,(\theta))\,d\left(\frac{1}{g'(\theta)}\right)$$

and hence

$$\left|\int_a^b r\,(g(\theta))\,d\theta\right| < \frac{\gamma}{g'(a)} + \frac{\gamma}{g'(b)} + \int_{\theta=a}^{b} \gamma \left|d\left(\frac{1}{g'(\theta)}\right)\right| \leqslant \frac{2\gamma}{g'(a)}$$

by the monotonicity of $g'(\theta)$.

Now let $\lambda_i^{(j)}$ $(i = 1, 2, \ldots, k_j)$ be the set of values of θ for which *either* $\{g_j(\theta)\} = \alpha$ *or* $\{g_j(\theta)\} = \beta$ $(j = 1, \ldots, l-1)$ i.e. the set of points of discontinuity of $r\,(g_j(\theta))$. Further let $\mu_1, \ldots, \mu_m$ be the numbers $0, 1, \lambda_i^{(j)}$ $(j = 1, \ldots, l-1)$ arranged in order of magnitude. Then the product

$$r\,(g_1(\theta))\,r\,(g_2(\theta)) \ldots r\,(g_{l-1}\,(\theta))$$

is constant and numerically less than 1 in each interval $\mu_n < \theta < \mu_{n+1}$. Hence

$$(7) \left\{\begin{aligned} \left|\int_0^1 r\,(g_1(\theta)) \ldots r\,(g_l(\theta))\,d\theta\right| &< \sum_{n=1}^{m-1} \left|\int_{\mu_n}^{\mu_{n+1}} r\,(g_l(\theta))\,d\theta\right| \leqslant 2\gamma \sum_{n=1}^{m-1} \frac{1}{g_l'(\mu_n)} \text{ (by (6))} \\ &\leqslant 2\gamma \left\{\frac{2l-1}{g_l'(0)} + \sum_{j=1}^{l-1} \sum_{i=3}^{k_j} \frac{1}{g_l'(\lambda_i^{(j)})}\right\}. \end{aligned}\right.$$

But

$$\int_{\lambda_{i-2}^{(j)}}^{\lambda_i^{(j)}} g_j'\,(\theta)\,d\theta = \int_{\theta=\lambda_{i-2}^{(j)}}^{\lambda_i^{(j)}} dg_j(\theta) = 1 \;(3 \leqslant i \leqslant k_j)$$

and hence

$$(8) \quad \sum_{i=3}^{k_j} \frac{1}{g_l'(\lambda_i^{(j)})} = \sum_{i=3}^{k_j} \int_{\lambda_{i-2}^{(j)}}^{\lambda_i^{(j)}} \frac{g_j'(\theta)}{g_l'(\lambda_i^{(j)})}\,d\theta \leqslant \sum_{i=3}^{k_j} \int_{\lambda_{i-2}^{(j)}}^{\lambda_i^{(j)}} \frac{g_j'(\theta)}{g_l'(\theta)}\,d\theta \leqslant 2\int_0^1 \frac{g_j'(\theta)}{g_l'(\theta)}\,d\theta.$$

The lemma follows on substituting (8) in (7).

We note also the [1])

Corollary. *Suppose* $g_1, \ldots, g_M$ *are any* M *of the functions* $f_1, f_2, \ldots$ *where* $M > l$

Then

$$(9) \left\{\begin{aligned} &\sum_{j_1 < j_2 < \ldots < j_l} \left|\int_0^1 r\,(g_{j_1})\,r\,(g_{j_2}) \ldots r\,(g_{j_l})\,d\theta\right| \\ &\leqslant 2\gamma \left\{(2l-1) \sum_{j=1}^{M} \frac{1}{g_j'(0)} + 2 \sum_{i<j\leqslant M} j^{l-2} \int_0^1 \frac{g_i'(\theta)}{g_j'(\theta)}\,d\theta\right\}. \end{aligned}\right.$$

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[1]) We suppress the argument θ except when its absence might cause ambiguity.

This follows directly by summation. We note that the right hand side of (9) is an increasing function of l (if $g_1, \ldots, g_M$ remain fixed).

4. *The principal lemmas.* The kernel of the proof lies in the next two lemmas.

Lemma 3. *Let α, β be any two numbers and let $\gamma = ||\beta-\alpha||$. Write $r(x)$ for $r(\alpha, \beta\colon x)$. Suppose that $g_1, \ldots, g_M$ are any M of the functions f_n and that*

$$M\gamma = M\,||\beta-\alpha|| \geqslant s$$

where s is a positive integer. Put

$$\mathfrak{r}(\theta) = \sum_{j=1}^{M} r(g_j(\theta))$$

and suppose

$$\mathfrak{S}\,(\text{say}) = (4s-1)\sum_{j=1}^{M}\frac{1}{g_j'(0)} + 2\sum_{i<j\leqslant M} j^{2s-2}\int_0^1 \frac{g_i'(\theta)}{g_j'(\theta)}\,d\theta \leqslant M.$$

Then

$$\int_0^1 \mathfrak{r}^{2s}(\theta)\,d\theta \leqslant 2\cdot\frac{(2s)^{2s+2}}{s!}\cdot(M\gamma)^s.$$

The proof depends on setting up an identity of the type

$$\text{(10)}\qquad \mathfrak{r}^{2s}(\theta) = D_0 + \sum_{l=1}^{2s} D_l \sum_{j_1<\ldots<j_l} r(g_{j_1})\,r(g_{j_2})\ldots r(g_{j_l})$$

by expanding and applying lemma 1, where $D_0, D_1, \ldots, D_{2s}$ are independent of θ and satisfy certain inequalities. The lemma will follow from the corollary to lemma 2, on integration.

In the first place we have an identity of the type

$$\mathfrak{r}^{2s}(\theta) = \sum_{\substack{n\\ a_1,\ldots,a_n}} B(a_1, \ldots, a_n) \sum_{j_1<\ldots<j_n} r^{a_1}(g_{j_1})\ldots r^{a_n}(g_{j_n})$$

where

(i) the first sum is over all sets of positive integers $n, a_1, \ldots, a_n$ with

$$\sum_{\nu=1}^{n} a_\nu = 2s$$

(ii) the numbers $B(a_1, \ldots, a_n)$ are non-negative integers and

$$\text{(11)}\qquad \sum_{\substack{n\\ a_1,\ldots,a_n}} B(a_1, \ldots, a_n) \leqslant \underbrace{(1+1\ldots+1)}_{2s\text{ summands}}{}^{2s} \leqslant (2s)^{2s}.$$

Further, $B(a_1, \ldots, a_n)$ is unchanged by permutation of $a_1, \ldots, a_n$.

Substituting from lemma 1, we obtain [1])

$$(12)\quad r^{2s}(\theta) = \sum_{\substack{l,m \\ b_1,\ldots,b_l,c_1,\ldots,c_m}} B(b_1,\ldots,b_l,c_1,\ldots,c_m)\, U_{b_1}\ldots U_{b_l} V_{c_1}\ldots V_{c_m} \sum_{\substack{j_1<\ldots<j_l \\ k_1<\ldots<k_m \\ (j_1,\ldots,j_l,k_1,\ldots,k_m)_{\neq}}} \gamma^m r(g_{j_1})\ldots r(g_{j_l}),$$

where l, m, $b_1, \ldots, b_l$, $c_1, \ldots, c_m$ are any set of numbers such that

$$\sum_{\lambda=1}^{l} b_\lambda + \sum_{\mu=1}^{m} c_\mu = 2s; \quad b_\lambda > 0,\ c_\mu > 0;\ l, m \geqslant 0.$$

Since $V_1 = 0$, we may assume that

$$c_\mu \geqslant 2 \quad (\mu = 1, \ldots, m)$$

and hence

$$m \leqslant s; \quad m \leqslant s-1 \quad \text{if } l > 0.$$

We now deduce the identity (10) where $D_0, \ldots, D_{2s}$ are independent of θ (but may depend on M, γ as well as s) and satisfy the inequalities

$$|D_0| \leqslant \frac{(2s)^{2s}}{s!}(M\gamma)^s \quad ; \quad |D_l| \leqslant \frac{(2s)^{2s}}{(s-1)!}(M\gamma)^{s-1} \quad (l \geqslant 1).$$

Indeed if D_0 is the sum of the terms in (12) independent of the r's, we have

$$|D_0| = \Big|\sum_{\substack{m \\ c_1,\ldots,c_m}} B(c_1, \ldots, c_m)\, V_{c_1} \ldots V_{c_m} \sum_{k_1<\ldots<k_m \leqslant M} \gamma^m\Big|$$

$$\leqslant \left|\sum_{\substack{m \\ c_1,\ldots,c_m}} B(c_1, \ldots, c_m) \frac{(M\gamma)^m}{m!}\right| \text{ since } |V_c| \leqslant 1,$$

$$\leqslant \Big|\sum_{\substack{m \\ c_1,\ldots,c_m}} B(c_1, \ldots, c_m)\Big| \frac{(M\gamma)^s}{s!} \text{ since } m \leqslant s,\ M\gamma \geqslant s,$$

$$\leqslant \frac{(2s)^{2s}}{s!}(M\gamma)^s \text{ by (11).}$$

Similarly the coefficient of a term $r(g_{j_1}) \ldots r(g_{j_l})$ is

$$D_l = \sum_{\substack{m \\ b_1,\ldots,b_l \\ c_1,\ldots,c_m}} B(b_1, \ldots, b_l, c_1, \ldots, c_m)\, U_{b_1} \ldots U_{b_l} V_{c_1} \ldots V_{c_m} \sum_{\substack{k_1<\ldots<k_m \leqslant M \\ (j_1,\ldots,j_l,k_1,\ldots,k_m)_{\neq}}} \gamma^m$$

which clearly depends only on l (and M, γ, s) and not on the individual numbers $j_1, \ldots, j_l$. Further, almost as for D_0,

$$|D_l| \leqslant \left|\sum_{\substack{m \\ b_1,\ldots,b_l \\ c_1,\ldots,c_m}} B(b_1, \ldots, b_l, c_1, \ldots, c_m) \frac{(M\gamma)^m}{m!}\right|$$

$$\leqslant \Big|\sum_{\substack{n \\ a_1,\ldots,a_n}} B(a_1, \ldots, a_n)\Big| \frac{(M\gamma)^{s-1}}{(s-1)!} \text{ since } M\gamma \geqslant s,\ m \leqslant s-1$$

$$\leqslant \frac{(2s)^{2s}}{(s-1)!}(M\gamma)^{s-1}.$$

[1]) The symbol $(x, y, \ldots, t)_{\neq}$ means that the numbers $x, y, \ldots, t$ are unequal in pairs.

Now integrate (10) and we obtain

$$\int_0^1 \mathfrak{r}^{2s}(\theta)\,d\theta \leqslant \frac{(2s)^{2s}}{s!}(M\gamma)^s + \frac{(2s)^{2s}}{(s-1)!}(M\gamma)^{s-1}\sum_{l=1}^{2s}\Big|\sum_{j_1<\ldots<j_l}\int_0^1 r(g_{j_1})\ldots r(g_{j_l})\,d\theta\Big|$$

$$\leqslant \frac{(2s)^{2s}}{s!}(M\gamma)^s + \frac{(2s)^{2s}}{(s-1)!}(M\gamma)^{s-1}\cdot 2s\cdot 2\gamma\mathfrak{S}$$

since, by the corollary to lemma 2, each of the terms of the outer summation is not greater than $2\mathfrak{S}$. Since $\mathfrak{S}\leqslant M$ by hypothesis, this proves the lemma.

Lemma 4. *Suppose $N > 100$ and suppose there is a positive integer $s \geqslant 4$ such that*

$$(13)\qquad \begin{cases} \mathfrak{S}_{a,M} = (4s-1)\sum_{j=1}^{M}\dfrac{1}{f'_{a+j}(0)} + 2\sum_{i<j\leqslant M} j^{2s-2}\displaystyle\int_0^1 \dfrac{f'_{a+i}(\theta)}{f'_{a+j}(\theta)}\,d\theta \\ \leqslant M \end{cases}$$

for all positive integers a, M with $a + M \leqslant N$, $M \geqslant N^{1/2}$. Then there is an absolute constant A_1 such that

$$\operatorname*{Max}_{n\leqslant N} \mathfrak{R}_N(\theta) \leqslant A_1\, s^{1/2} N^{1/2} \log^{p/s} N$$

except, possibly, in a set E of θ of measure

$$(14)\qquad |E| \leqslant 4\log^{-2p} N,$$

where p is any positive number.

Choose U, V integers such that

$$2^{2U} \leqslant s\,N < 2^{2U+2}\quad,\quad 2^{2V+2}\leqslant s^{-1}N < 2^{2V+4}.$$

We shall show first that there is a set E for which (14) holds and

$$(15)\qquad \operatorname*{Max}_{\substack{u=0,1,\ldots,[N_2-U]\\ v=0,1,\ldots,2^V-1}} |R_{u2^U}(0, v2^{-V}:\theta)| \leqslant A\, s^{1/2} N^{1/2}\log^{p/s} N$$

except, possibly, when $\theta\in E$.

Let a, y, b, z be any four integers with

$$y \geqslant U\ ;\ 0\leqslant a2^y < (a+1)\,2^y \leqslant N$$
$$0 < z \leqslant V\ ;\ 0 \leqslant b2^{-z} < (b+1)\,2^{-z} \leqslant 1.$$

Then

$$\mathfrak{r}(\theta) = \mathfrak{r}_{a,y;\,b,z}(\theta) = {}_{a2^y}R_{(a+1)2^y}(b2^{-z}, \overline{b+1}\,2^{-z}:\theta)$$
$$= \sum_{j=1}^{2^y} r(f_{a2^y+j}(\theta)),$$

where $r(x) = r(b\,2^{-z}, \overline{b+1}\,2^{-z}: x)$ satisfies the conditions of lemma 3 by (13), and so

$$\int_0^1 \mathfrak{r}^{2s}(\theta) \leqslant 2\cdot\frac{(2s)^{2s+2}}{s!}\,2^{s(y-z)}.$$

Hence

$$|\mathfrak{r}_{a,y;\,b,z}(\theta)| \leqslant As^{1/2} \cdot \log^{p/s} N \cdot N^{1/4}\, 2^{1/4(y-z)}$$

except, possibility, in a set $E_{a,y;\,b,z}$ of θ of measure

$$|E_{a,y;\,b,z}| \leqslant (\log N)^{-2p} \cdot N^{-s/2}\, 2^{s/2(y-z)}.$$

We shall take for the E of the theorem

$$E = \bigcup E_{a,y;\,b,z}.$$

Then certainly

$$|E| \leqslant \sum |E_{a,y;\,b,z}| \leqslant (\log N)^{-2p} \left(\sum_{a,y} N^{-s/2}\, 2^{sy/2}\right) \left(\sum_{b,z} 2^{-(sz/2)}\right).$$

Now for any given value of z there are 2^z values of b and hence

$$\sum_{b,z} 2^{-sz/2} = \sum_{z} 2^{-\left(\frac{s-2}{2}\right)z} \leqslant 2 \qquad (s \geqslant 4).$$

Similarly for any value of y there are at most $[N\, 2^{-y}]$ values of a and hence

$$\sum_{a,y} N^{-s/2}\, 2^{sy/2} \leqslant \sum_{2^y \leqslant N} (2^y N^{-1})^{\,1-(s/2)} \leqslant 2 \qquad (s \geqslant 4).$$

Hence, finally,

$$|E| \leqslant 4\, (\log N)^{-2p}.$$

For the rest of the proof of this lemma we suppose $\theta \notin E$ and suppress the argument θ. Now consider a general

$$R_{u\,2^U}(0,\, v\, 2^{-V}).$$

By expressing u and v in the binary scale and making use of the basic identities

$$_{N_1}R_{N_3} = {}_{N_1}R_{N_2} + {}_{N_2}R_{N_3} \;\; ; \;\; R(\alpha, \beta) = R(\alpha, \gamma) + R(\gamma, \beta)$$

we have

$$R_{u\,2^U}(0,\, v\, 2^{-V}) = \sum{}^{*}\, \mathfrak{r}_{a,y;\,b,z}$$

where the * indicates a sum depending on u and v in which each pair of values y, z occurs at most once. Hence

$$|R_{u\,2^U}(0, v\, 2^{-V})| \leqslant As^{1/2} \log^{p/s} N \sum_{\substack{2^y \leqslant N \\ z > 0}} N^{1/4}\, 2^{1/4(y-z)} \leqslant As^{1/2} \log^{p/s} N \cdot N^{1/2}$$

as required.

We now complete the proof of the lemma. If $0 \leqslant n \leqslant N$ we can find a $u\, 2^U$ such that $u\, 2^U \leqslant N \leqslant (u+1)\, 2^U \leqslant u\, 2^U + s^{1/2} N^{1/2}$. Hence from (15) and the trivial inequalities $|R_{a+b}| \leqslant |R_a| + |{}_aR_{a+b}| \leqslant |R_a| + b$ we have

$$|R_n(0,\, v\, 2^{-V})| \leqslant As^{1/2} N^{1/2} \log^{p/s} N + s^{1/2} N^{1/2} \leqslant As^{1/2} N^{1/2} \log^{p/s} N.$$

Next, if v, w are any two integers in the range $0 \leqslant v, w < 2^V$ we have

$|R_n(w\,2^{-V},\ v\,2^{-V})| \leqslant |R_n(0, w\,2^{-V})| + |R_n(0, v\,2^{-V})| \leqslant As^{1/2} N^{1/2} \log^{p/s} N.$

In particular

$$|R_n(\delta_1, \delta_1 + 2^{-V})| \leqslant As^{1/2} N^{1/2} \log^{p/s} N$$

if δ_1 is of the form $v\,2^{-V}$. Suppose now $\{\delta - \delta_1\} < 2^{-V}$. Then, by definition,

$$R_n(\delta_1, \delta) = F_n(\delta_1, \delta) - n\{\delta - \delta_1\}$$
$$R_n(\delta_1, \delta_1 + 2^{-V}) = F_n(\delta_1, \delta_1 + 2^{-V}) - n\,2^{-V}$$
$$0 \leqslant F_n(\delta_1, \delta) \leqslant F_n(\delta_1, \delta_1 + 2^{-V}),$$

where

$$0 \leqslant n\{\delta - \delta_1\} \leqslant n\,2^{-V} \leqslant N\,2^{-V} \leqslant 4\,s^{1/2} N^{1/2}.$$

Hence

$$|R_n(\delta_1, \delta)| \leqslant |R_n(\delta_1, \delta_1 + 2^{-V})| + 4\,s^{1/2} N^{1/2} \leqslant As^{1/2} N^{1/2} \log^{p/s} N.$$

Finally, if α, β are any numbers we can find α_1, β_1 of the form $v\,2^{-V}$ such that $\{\alpha - \alpha_1\} < 2^{-V}$, $\{\beta - \beta_1\} < 2^{-V}$. Then

$$|R_n(\alpha, \beta)| \leqslant |R_n(\alpha_1, \alpha)| + |R_n(\beta_1, \beta)| + |R_n(\alpha_1, \beta_1)| \leqslant As^{1/2} N^{1/2} \log^{p/s} N$$

for all n, α, β. Since

$$\mathfrak{R}_n = \operatorname*{Max}_{\alpha, \beta} |R_n(\alpha, \beta)|$$

this proves the lemma.

5. *An elementary lemma.* Before going to the proof of the theorem we give an almost trivial lemma. We again suppress the argument θ.

Lemma 5. *Let the sequence of functions*

(16) $$f_1, \ldots, f_N$$

be decomposed into a number (say t) of distinct subsequences:

(17) $$\begin{cases} f_1^{(1)}, f_2^{(1)}, \ldots, f_{N_1}^{(1)} \\ \quad\vdots \\ f_1^{(t)}, f_2^{(t)}, \ldots, f_{N_t}^{(t)} \end{cases}$$

so that every element in (16) *occurs just once in* (17) *and vice-versa. Suppose also that the elements of any row of* (17) *occur in the same order in* (16). *Then*

$$\operatorname*{Max}_{n \leqslant N} \mathfrak{R}_n(\theta) \leqslant \operatorname*{Max}_{n \leqslant N_1} \mathfrak{R}_n^{(1)}(\theta) + \ldots + \operatorname*{Max}_{n \leqslant N_t} \mathfrak{R}_n^{(t)}(\theta)$$

where the upper affixes in the $\mathfrak{R}_n^{(\tau)}$ refer to the sequence $f_n^{(\tau)}$.

For we have

$$R_n(\alpha,\beta)=\sum_{\nu=1}^{n} r(\alpha,\beta:f_\nu)=\sum_{\tau=1}^{t}\sum_{\nu_\tau=1}^{n_\tau} r(\alpha,\beta:f^{(\tau)}_{\nu_\tau})=\sum_{\tau=1}^{t} R^{(\tau)}_{n_\tau}(\alpha,\beta)$$

for some integers n_τ. Hence

$$\operatorname*{Max}_{n\leqslant N}\mathfrak{R}_n=\operatorname*{Max}_{\substack{\alpha,\beta\\ n\leqslant N}}|R_n(\alpha,\beta)|\leqslant\sum_{\tau=1}^{t}\operatorname*{Max}_{\substack{\alpha,\beta\\ n_\tau\leqslant N_\tau}}|R^{(\tau)}_{n_\tau}(\alpha,\beta)|=\sum_{\tau=1}^{t}\operatorname*{Max}_{n_\tau\leqslant N_\tau}\mathfrak{R}^{(\tau)}_{n_\tau}.$$

6. *Proof of the theorem.*

We first adapt lemma 4.

Lemma 6. *Suppose the conditions of the theorem are satisfied. Then there is an absolute constant A_2 and a constant C_0 depending only on the function $\varphi(n)$ such that for any given $N > C_0$ the inequality*

$$\operatorname*{Max}_{n\leqslant N}\mathfrak{R}_N(\theta)\leqslant A_2\, n^{1/2}\log^{1/2}N\log\log N\,\varphi^{-1/2}(N)$$

holds except, possibly, in a set E_N of θ of measure

$$|E_N|\leqslant\log^{-2}N.$$

We shall denote by C a constant depending only on the function $\varphi(n)$, not necessarily the same in different contexts (so c of (ii) of the theorem is a C). Put

$$p=3 \tag{18}$$

and let $s=s_N$ and $t=t_N$ be the integers

$$s=[\log\log N]+1,\quad t=\left[\frac{6s\log N}{\varphi(N)}\right]+1. \tag{19}$$

We may assume that N is so large that

$$s\geqslant 4,\quad t\leqslant\log^2 N \tag{20}$$

by condition (ii) of the theorem.

The proof depends on decomposing the sequence $f_1,\ldots,f_N$ into t subsequences, in the sense of lemma 5, and then applying lemma 4 to the subsequences.

Consider the t subsequences

$$f^{(\tau)}_1,\ldots,f^{(\tau)}_{N_\tau}\quad(\tau=1,\ldots,t)$$

where

$$f^{(\tau)}_n=f_{(n-1)t+\tau},\quad |N_\tau-(N/t)|\leqslant 1.$$

We now estimate the sum $\mathfrak{S}^{(\tau)}_{a,M}$ $(M\geqslant N_\tau^{1/2})$ of lemma 4 for each of the subsequences $f^{(\tau)}_n$:

$$\mathfrak{S}^{(\tau)}_{a,M}=(4s-1)\sum_{j=1}^{M}\frac{1}{f'^{(\tau)}_{a+j}(0)}+2\sum_{i<j\leqslant M} j^{2s-2}\int_0^1\frac{f'^{(\tau)}_{a+i}(\theta)}{f'^{(\tau)}_{a+j}(\theta)}\,d\theta$$
$$=(4s-1)\,\sigma_1+2\,\sigma_2\ \text{(say)}.$$

Now by (i) and (ii) of the theorem,

$$\sum_{n=1}^{\infty} \frac{1}{f''_n(\theta)} \leqslant 1 + \sum_{n=2}^{\infty} e^{-\varphi(2)\ldots-\varphi(n)} < C < \infty$$

and *a fortiori*

$$\sigma_1 < C.$$

Also, if $0 < i < j$,

$$\frac{f'^{(\tau)}_{a+i}(\theta)}{f'^{(\tau)}_{a+j}(\theta)} = \frac{f'^{(\theta)}_{(a+i-1)\,t+\tau}}{f'^{(\theta)}_{(a+j-1)\,t+\tau}} \leqslant e^{-\varphi(\overline{a+j-1}\,t+\tau)-\ldots-\varphi(\overline{a+j-2}\,t+\tau+1)}$$

$$\leqslant e^{-t\varphi(j)}$$

since $\varphi(n)$ is non-decreasing and $\overline{a+j-2}\,t+\tau+1 = (a+j-2)(t-1)+ + (\tau-1)+a+j \geqslant j$. Hence

$$\sigma_2 \leqslant \sum_{j \leqslant M} j^{2s} e^{-t\varphi(j)}$$

$$\leqslant \sum_{\log j \leqslant \sqrt{\log N}} j^{2s} + \sum_{\substack{\log j > \sqrt{\log N} \\ j \leqslant M}} e^{2s \log j - t\varphi(j)}$$

$$= \sigma_3 + \sigma_4 \text{ (say).}$$

But trivially $\sigma_3 < N^{1/3}$ if N is large enough. Further

$$t\,\varphi(j) \geqslant t\,\varphi(N) \log^{-1} N \log\log^{-1} N \log j \log\log j \geqslant 6 \log j \log\log j$$

by the monotonicity of $\log n \log\log n\, \varphi^{-1}(n)$ and hence

$$\sigma_4 \leqslant \sum_{\log j \geqslant \sqrt{\log N}} e^{2s \log j - 6 \log j \log\log j}$$

$$\leqslant \sum e^{-\log j \log\log j} < A < \infty$$

since $2s = 2[\log\log N] + 2 < 5 \log\log j$ if $\log j \geqslant \log N$. Combining these inequalities we deduce

$$(21) \qquad \mathfrak{S}^{(\tau)}_{a,M} \leqslant 4s\sigma_1 + 2\sigma_3 + 2\sigma_4 < Cs + 2N^{1/3} + A < \left(\frac{N}{t} - 1\right)^{1/2} \leqslant N_\tau^{1/2}$$

for N greater than a C, the third inequality in (21) being a trivial deduction from (19). Hence $\mathfrak{S}^{(\tau)}_{a,M} \leqslant M$ for all $M \geqslant N_\tau^{1/2}$ and, by lemma 4 applied to $f^{(\tau)}_n$ it follows that

$$\operatorname*{Max}_{n \leqslant N_\tau} \mathfrak{R}^{(\tau)}_n(\theta) \leqslant A s^{1/2} N_\tau^{1/2} \log^{p/s} N_\tau$$

except, possibly, in a set $E^{(\tau)}_N$ (say) of measure

$$|E^{(\tau)}_N| \leqslant 4 \log^{-2p} N_\tau.$$

Write

$$E_N = \bigcup_\tau E^{(\tau)}_N.$$

Then in the first place

$$|E_N| \leqslant 4t \log^{-2p}\left(\frac{N}{t} - 1\right)$$

$$\leqslant \log^{-2} N$$

by (18) and (20) if N is greater than a C. Also, provided $\theta \,\overline{\in}\, E_N$ we have

$$\begin{aligned}\operatorname*{Max}_{n \leqslant N} \mathfrak{R}_N(\theta) &\leqslant \sum_{\tau=1}^{t} \operatorname*{Max}_{n_\tau \leqslant N_\tau} \mathfrak{R}^{(\tau)}_{n_\tau}(\theta)\\ &\leqslant A\, t\, s^{1/2}\left(\frac{N}{t}+1\right)^{1/2} \log^{p/s} N\\ &\leqslant A\, t^{1/2} s^{1/2} N^{1/2} \log^{p/s} N\end{aligned}$$

since each $N_\tau \leqslant (N/t) + 1 \leqslant N$. Hence using the values (18), (19) we have

$$\operatorname*{Max}_{n \leqslant N} \mathfrak{R}_n(\theta) \leqslant A\, N^{1/2} \log^{1/2} N \log\log N\, \varphi^{-1/2}(N)$$

which proves the lemma.

It is now an easy matter to prove the theorem. The proof follows familiar lines.

We shall first prove that for almost all θ there is a $T_0(\theta)$ such that

$$\operatorname*{Max}_{N \leqslant 2^T} \mathfrak{R}_N(\theta) \leqslant A_2\, 2^{1/2\, T} \log^{1/2} 2^T \log\log 2^T\, \varphi^{-1/2}(2^T)$$

for all $T \geqslant T_0(\theta)$ where A_2 is the A_2 of lemma 6. For if E_N is the E_N of lemma 6, that lemma shows that

$$\sum_T |E_{2^T}|$$

is convergent. Hence for almost all θ there is a $T_0(\theta)$ such that

$$\theta \,\overline{\in}\, E_{2^T} \quad \text{all} \quad (T \geqslant T_0).$$

Now put $N_0(\theta) = 2^{T_0(\theta)}$. Then for all $N \geqslant N_0(\theta)$ there is a $T \geqslant T_0(\theta)$ such that $N \leqslant 2^T \leqslant 2N$. But then

$$\mathfrak{R}_N(\theta) \leqslant \operatorname*{Max}_{N \leqslant 2^T} \mathfrak{R}_N(\theta) \leqslant A\,(2N)^{1/2} \cdot \log^{1/2} 2N \cdot \log\log 2N\, \varphi^{-1/2}(N)$$

since $\varphi(n)$ is non-decreasing, and finally

$$\mathfrak{R}_N(\theta) \leqslant A\, N^{1/2} \log^{1/2} N \log\log N\, \varphi^{-1/2}(N)$$

for some A. This proves the theorem.

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