MATHEMATICS

THE ASYMPTOTIC EXPANSION OF THE CONFLUENT HYPERGEOMETRIC FUNCTION $M_{\omega/2,0}(2\omega)$

BY

H. A. LAUWERIER

(Communicated by Prof. J. G. VAN DER CORPUT at the meeting of December 17, 1949)

§ 1. Elsewhere [1]) I have found that $M_{\omega/2,0}(2\omega)$ is for large positive ω asymptotic equal to

$$2^{1/3}\, 3^{-1/6}\, \pi^{-1}\, \omega^{1/6} \{\Gamma(^1/_3) \cos(\omega/2 - {}^1/_6)\, \pi \sum_0 a_p\, \omega^{-2p} + \\ + 2.3^{1/2}\, \Gamma(^2/_3) \cos(\omega/2 + {}^1/_6) \sum_0 b_p\, \omega^{-2p-4/3}\}.$$

Here

$$a_0 = 1 \;;\; b_0 = -\,{}^{11}/_{280} \;;\; a_1 = -\,{}^{13}/_{900} \;;\; b_1 = {}^4/_7 + {}^1/_9 + {}^2/_{13} - {}^7/_{32} - {}^{377}/_{625}.$$

The object of this paper is to deduce a numerical upper bound of the remainder in this formula.

§ 2. We start with the fundamental formula

$$M_{\omega/2,0}(2\omega) = (2\omega)^{1/2}\, 1/2\pi i \int_{-\infty}^{(1+)} e^{\omega z} (z - 1/z + 1)^{\omega/2} (z^2 - 1)^{-1/2}\, dz.$$

We put $\arg(z-1) = 0$ and $\arg(z+1) = 0$ for $z > 1$. We may write

$$(2\omega)^{-1/2} M_{\omega/2,0}(2\omega) = -\, 1/\pi\, Re\, \{\exp(\omega\pi i/2) \int_a^{-\infty} e^{\omega\varphi(z)} (1 - z^2)^{-1/2}\, dz\}.$$

Here a denotes an arbitrary number > 1 and the path of integration lies above the real axis; $\varphi(z) = z + {}^1/_2 \ln 1 - z/1 + z$.

According to the method of steepest descent we define the path of integration by $Im\, \varphi(z) = 0$, thus if $z = x + iy$,

$$\begin{cases} x^2 = 2y \operatorname{cotg} 2y + 1 - y^2 & 0 < y < \pi/2 \\ y = 0 & 0 < x < 1. \end{cases}$$

Therefore the contour consists of a segment S of the real axis from $x = 1$ to $x = 0$ and moreover of a curve L in the second quadrant with asymptote $y = \pi/2$ and with the tangent $y = x \operatorname{tg} 2\pi/3$ at the origin. The

---

[1]) The use of confluent hypergeometric functions in mathematical physics and the solution of an eigenvalue problem. Appl. Sci. Res. (A) 1950.

function $\varphi(z)$ increases on S monotonously from $-\infty$ to 0 and decreases on L monotonously from 0 to $-\infty$.

By the transformation $\zeta = \{-\varphi(z)\}^{1/3}$, where $\arg \zeta = 0$ for z on S and $\arg \zeta = 2\pi/3$ for z on L, we find for $\pi (2\omega)^{-1/2} M_{\omega/2, 0}(2\omega)$ the expression

$$Re\, e^{\omega\pi i/2} \left\{ \int_0^\infty e^{-\omega\zeta^3} \frac{d\zeta}{d\zeta/dz\,(1-z^2)^{1/2}} - \int_0^{\infty \exp 2\pi i/3} e^{-\omega\zeta^3} \frac{d\zeta}{d\zeta/dz\,(1-z^2)^{1/2}} \right\}.$$

By applying BURMANN's theorem we obtain

$$\pi (2\omega)^{-1/2} M_{\omega/2, 0}(2\omega) = Re\, e^{\omega\pi i/2} \{ \sum_0^{n-1} m_k (A_k - B_k) + (C_n - D_n) \},$$

where

$$(1) \quad \begin{cases} A_k = \int_0^\infty e^{-\omega\zeta^3} \zeta^k \, d\zeta = {}^1/_3\, \Gamma(k + 1/3)\, \omega^{-(k+1/3)}; \\ B_k = \int_0^{\infty \exp 2\pi i/3} e^{-\omega\zeta^3} \zeta^k \, d\zeta = {}^1/_3\, \Gamma(k+1/3)\, \omega^{-(k+1/3)\pi i}\, e^{(2k+2/3)\pi i}; \\ m_k = 1/2\pi i \int^{(0+)} (1-t^2)^{-1/2} (-t - {}^1/_2 \ln 1-t/1+t)^{-(k+1/3)}\, dt; \end{cases}$$

$$(2) \quad \begin{cases} C_n = \int_0^\infty e^{-\omega\zeta^3} \zeta^n R_n(\zeta)\, d\zeta \quad ; \quad D_n = \int_0^{\infty \exp 2\pi i/3} e^{-\omega\zeta^3} \zeta^n R_n(\zeta)\, d\zeta; \\ R_n(\zeta) = 1/2\pi i \int \frac{dt}{(1-t^2)^{1/2} (-t - {}^1/_2 \ln 1-t/1+t)^{n/3} \{(-t - {}^1/_2 \ln 1-t/1+t)^{1/3} - \zeta\}}; \end{cases}$$

The contour encloses all the values of t for which $(-t - {}^1/_2 \ln 1-t/1+t)^{1/3} = \zeta$. It is easily seen that $m_k = 0$ for k odd so that in virtue of the vanishing of the factor $\sin (k + 1/3)\pi$ for $k \equiv 2 \pmod 3$ we find

$$M_{\omega/2, 0}(2\omega) = \frac{2^{1/2} \omega^{1/6} \cos(\omega/2 - {}^1/_6)\pi}{3^{1/2}\pi} \sum_0^{[n-1/6]} m_{6p}\, \Gamma(2p + {}^1/_3)\, \omega^{-2p} +$$

$$+ \frac{2^{1/2} \omega^{-1/6} \cos(\omega/2 + {}^1/_6)\pi}{3^{1/2}\pi} \sum_0^{[n-5/6]} m_{6p+4}\, \Gamma(2p + 1 + {}^2/_3)\, \omega^{-2p-1} + U_n,$$

where the remainder U_n is equal to

$$(3) \qquad U_n = (2\omega)^{1/2}/\pi \, \{ C_n \cos(\omega\pi/2) - Re\,(D_n e^{\omega\pi i/2}) \}.$$

In the paper mentioned in note [1]), in which I have derived the same asymptotic expansion, I have found

$$m_0 = 3^{1/3} \quad ; \quad m_4 = -\, 11.3^{5/3}/280 \quad ; \quad m_6 = -\, 13.3^{1/3}/400;$$
$$m_{10} = ({}^4/_7 + {}^1/_9 + {}^2/_{13} - {}^7/_{32} - {}^{377}/_{625})\, 3^{11/3}/40.$$

§ 3. In this section I deduce for k even an upper bound for the absolute value of m_k, viz.

$$(4) \qquad |m_k| < s\, (2/\pi)^{\frac{k+4}{3}} \left\{ 0{\cdot}64 + \frac{3.2^{-\frac{k+1}{6}}}{k+1} \right\},$$

where

$$s = 1 \quad \text{if} \quad k \equiv 0 \text{ or } 4 \pmod 6$$
$$s = 2 \quad \text{if} \quad k \equiv 2 \pmod 6.$$

In (2) I choose as path of integration two contours resp. around $+1$ and -1 and beginning and also ending resp. at $+\infty$ and $-\infty$. In this manner I obtain

$$m_k = (1/\pi)\, Re\, (\exp\, (k + 1/3)\, \pi i - 1) \int_1^\infty (t^2 - 1)^{-1/2} \left(\theta + \frac{\pi i}{2}\right)^{-(k+1/3)} dt,$$

where $\theta = t + {}^1/_2 \ln t - 1/t + 1$, and therefore we obtain

$$|m_k| \leqq s/\pi \int_1^\infty (t^2 - 1)^{-1/2} \left(\theta^2 + \frac{\pi^2}{4}\right)^{-(k+1/6)} dt.$$

By using

$$\theta^2 + (\pi^2/4) \geqq (\pi^2/4) \qquad \text{for} \quad 1 < t < a$$
$$\theta^2 + (\pi^2/4) \geqq (t - b)^2 + (\pi^2/4) \quad \text{for} \quad t > a,$$

where a is an arbitrary number $> {}^5/_4$ and $b = {}^1/_2 \ln a - 1/a + 1$, we find

$$(5) \quad \begin{cases} |m_k| < s/\pi\, (2/\pi)^{k+1/3} \{\ln (a + \sqrt{a^2 - 1}) + \\ + \int_c^\infty (u^2 + 1)^{-(k+1/6)} \{u^2 + (4/\pi)\, bu - (4/\pi^2)\, (b^2 - 1)\}^{-1/2} du\}, \end{cases}$$

where

$$c = 2/\pi\, (a - b) = 2/\pi\, (a - {}^1/_2 \ln a + 1/a - 1).$$

Here

$$u^2 + (4/\pi)\, bu - 4/\pi^2\, (b^2 - 1) > u^2.$$

In fact we have to prove the inequality

$$2\, b\, (a - b) > b^2 - 1,$$

which is equivalent to

$$3\, b^2 - 2\, ab - 1 < 0,$$

that is

$$b < \frac{a + \sqrt{a^2 + 3}}{3};$$

this inequality follows from

$$a + \sqrt{a^2 + 3} - {}^3/_2 \ln (a + 1/a - 1) > 0$$

which is obvious since the left hand side represents a monotonously increasing function which is positive at $a = {}^5/_4$.

Consequently the integral occurring in (5) is less than

$$\int_c^\infty u^{-1}(u^2+1)^{-(k+1/6)}\,du = {}^1/_2 \int_{1+c^2}^\infty v^{-(k+7/6)}/1-(1/v)\,dv = {}^1/_2 \sum_0^\infty \int_{1+c^2}^\infty v^{-((k+7/6)+j)}\,dv$$

$$= {}^1/_2 \sum_0^\infty \frac{1}{(1+c^2)^{(k+7/6)+j}\,((k+1/6)+j)} \leqq$$

$$\leqq \frac{3}{k+1}\sum_0^\infty \frac{1}{(1+c^2)^{(k+1/6)+j}} = \frac{3}{(k+1)\,c^2\,(1+c^2)^{k-5/6}}.$$

We define a by the equation $c=1$. In that case $a<1{,}94$ and $\ln(a+\sqrt{a^2-1})<1{,}28$. This establishes the proof of formula (4).

§ 4. The object of the following sections is to deduce an upper bound for the expression $R_n(\zeta)$ occurring in (2). We choose the same path of integration as in the preceding section, namely two contours resp. around $+1$ and -1, and beginning and also ending resp. at $+\infty$ and $-\infty$. Thus we find for $2\pi R_n(\zeta)$ the sum

$$\int_1^\infty \frac{\exp(n+1/3)\,dt}{(t^2-1)^{1/2}(\theta+(\pi i/2))^{n/3}\{(\theta+(\pi i/2))^{1/3}-\zeta e^{\pi i/3}\}} + \int_1^\infty \frac{\exp-(n+1/3)\,\pi i\,dt}{(t^2-1)^{1/2}(\theta-(\pi i/2))^{n/3}\{(\theta-(\pi i/2))^{1/3}-\zeta e^{\pi i/3}\}}$$

$$-\int_1^\infty \frac{dt}{(t^2-1)^{1/2}(\theta+(\pi i/2))^{n/3}\{(\theta+(\pi i/2))^{1/3}-\zeta\}} - \int_1^\infty \frac{dt}{(t^2-1)^{1/2}(\theta-(\pi i/2))^{n/3}\{(\theta-(\pi i/2))^{1/3}-\zeta\}}$$

where $\theta = t + {}^1/_2 \ln(t-1/t+1)$.

This is in absolute value at most

$$\int_1^\infty \frac{1}{(t^2-1)^{1/2}\,|\theta+(\pi i/2)|^{n/3}}\left|\frac{\exp(n+1/3)\,\pi i}{(\theta+(\pi i/2))^{1/3}-\zeta e^{\pi i/3}} - \frac{1}{(\theta+(\pi i/2))^{1/3}-\zeta}\right| dt +$$

$$+\int_1^\infty \frac{1}{(t^2-1)^{1/2}\,|\theta-(\pi i/2)|^{n/3}}\left|\frac{\exp-(n+1/3)\,\pi i}{(\theta-(\pi i/2))^{1/3}-\zeta e^{-\pi i/3}} - \frac{1}{(\theta-(\pi i/2))^{1/3}-\zeta}\right| dt.$$

Hence

$$|R_n(\zeta)| \leqq (1/\pi)\int_1^\infty \frac{T\,dt}{(t^2-1)^{1/2}(\theta^2+(\pi^2/4))^{n/6}\,|(\theta+(\pi i/2))^{1/3}-\zeta e^{\pi i/3}|\,|(\theta+(\pi i/2))^{1/3}-\zeta|}, \tag{6}$$

where

$$T = (\theta^2+(\pi^2/4))^{1/6} + 3^{1/2}|\zeta| \quad \text{if} \quad n \equiv 4 \pmod 6$$

$$T = (\theta^2+(\pi^2/4))^{1/6} \quad \text{if} \quad n \equiv 0 \pmod 6.$$

We consider only values of n which are either $\equiv 0 \pmod 6$ or $\equiv 4 \pmod 6$.

§ 5. In this section I shall prove the inequality

$$(7)\quad |(\theta+(\pi i/2))^{1/3}-\zeta e^{\pi i/3}|\,|(\theta+(\pi i/2))^{1/3}-\zeta| \geqq \frac{(\pi/2)^{2/3}}{1+h_1\zeta} \text{ if } \arg\zeta=0,$$

where

$$h_1=(2+\sqrt{3})\,(2/\pi)^{1/3}=3{\cdot}2105;$$

and

$$(8)\quad |(\theta+(\pi i/2))^{1/3}-\zeta e^{\pi i/3}|\,|(\theta+(\pi i/2))^{1/3}-\zeta| \geqq |\theta+(\pi i/2)|^{2/3} \text{ if } \arg\zeta=(2\pi/3).$$

For $t \geqq 1$ the function θ runs though all real values, so that the point $(\theta+(\pi i/2)^{1/3}=X+i\,Y$ lies on the curve F defined by

$$Im\,(X+i\,Y)^3=\pi/2 \quad \text{i.e.} \quad 3\,X^2Y-Y^3=\pi/2.$$

This curve has the symmetry axis $Y=X\,3^{-1/2}$ and two asymptotes $Y=0$ and $Y=X\,3^{1/2}$. Now we distinguish two cases.

1. Put $\arg\zeta=0$. Choosing a new rectangular coordinate system the x-axis of which coincides with the symmetry axis of F, we find for the equation of F

$$x^3-3\,xy^2=\pi/2$$

that is

$$x=(\pi/2)^{1/3}\,\lambda \qquad y=3^{-1/2}\,(\pi/2)^{1/3}\,\lambda^{-1/2}\,(\lambda^3-1)^{1/2}.$$

With respect to this system the coordinates of the point $\zeta e^{\pi i/3}$ are $({}^1/_2\,\zeta\sqrt{3},\ {}^1/_2\,\zeta)$ and those of ζ are $({}^1/_2\,\zeta\sqrt{3},\ -{}^1/_2\,\zeta)$. By putting $\zeta=2.3^{-1/2}\,(\pi/2)^{1/3}\mu$, we obtain

$$|(\theta+(\pi i)2))^{1/3}-\zeta e^{\pi i/3}|^2\,|(\theta+(\pi i/2))^{1/3}-\zeta|^2=$$
$$=\frac{(\pi/2)^{4/3}}{9\,\lambda^2}\,\{9\,\lambda^2\,(\lambda-\mu)^4+6\,\lambda\,(\lambda-\mu)^2\,(\lambda^3+\lambda\mu^2-1)+(\lambda^3-\lambda\mu^2-1)^2\}.$$

The substitution

$$v=4\,\lambda^2\,(\lambda-\mu) \quad w=4\,\lambda\,(\lambda-\mu)^2$$

transforms the right hand side into

$$\frac{(\pi/2)^{4/3}}{9\,\lambda^2}\,(v^2-vw+w^2-v-w+1).$$

In virtue of $\lambda \geqq 1$ and $\mu \geqq 0$ we obtain $v^2 \geqq 4\,w \geqq 0$ and $v\,(v-w) \geqq 0$.

In the next section we shall prove that these inequalities imply

$$(9)\qquad v^2-vw+w^2-v-w+1 \geqq \frac{9\,v^2}{\{f\,(v-w)+(4\,vw)^{1/2}\}^2},$$

where $f=2+4.3^{-1/2}=4.309$.

Hence the left hand side of (7) is at most equal to $(\pi/2)^{2/3}\,(1+h_1\zeta)^{-1}$ where h_1 has the above indicated value.

2. Put $\arg\zeta = 2\pi/3$. The square of the left hand side of (8) is equal to

$$(X^2 + Y^2 + 2\,|\zeta|\,X + |\zeta|^2)\ (X^2 + Y^2 + |\zeta|\,X - |\zeta|\,Y\sqrt{3} + |\zeta|^2) =$$
$$= (X^2 + Y^2)^2 + |\zeta|\,(X^2 + Y^2 + |\zeta|^2)\ (X\sqrt{3} - Y)\sqrt{3} +$$
$$+ 2\,|\zeta|^2\,(2\,X^2 - XY\sqrt{3} + Y^2) + |\zeta|^4 \geqq (X^2 + Y^2)^2,$$

since each point of F satisfies the inequality $Y \leqq X\sqrt{3}$. This proves inequality (8).

§ 6. In this section the inequality (9) will be proved. We distinguish four cases, namely

1. $v \leqq 0$; 2. $v \geqq 0$ $w \geqq 4$; 3. $v \geqq 0$, $0 \leqq w \leqq (2 - \sqrt{3})^2$
4. $v \geqq 0$, $(2 - \sqrt{3})^2 \leqq w \leqq 4$.
1. The left hand side of (9) is $\geqq {}^3/_4$ and the right hand side of (9) is $\leqq {}^9/_{16}$.
2. Writing $v = w + \delta$ $\delta \geqq 0$ we have

$$(v^2 - vw + w^2 - v - w + 1)^{1/2} = \{(w - 1)^2 + \delta\,(w - 1) + \delta^2\}^{1/2} \geqq w - 1\ ;$$

$$\frac{3\,v}{f\,(v - w) + (4\,vw)^{1/3}} \leqq \frac{3\,(w + \delta)}{f\,\delta + (4\,w^2)^{1/3}} \leqq \frac{3\,w^{1/3}}{2^{2/3}}$$

(9) follows from

$$w - 1 \geqq 3.2^{-2/3}\,w^{1/3},$$

which is obvious.

3. For $0 \leqq w \leqq (2 - \sqrt{3})^2$ we have

$$(v^2 - vw + w^2 - v - w + 1)^{1/2} = \{(v - {}^1/_2\,w - {}^1/_2)^2 + {}^3/_4\,(1 - w)^2\}^{1/2} \geqq$$
$$\geqq (\sqrt{3}/2)\,(1 - w) \geqq 3\,(2 - \sqrt{3});$$

and, writing $v = 2\,\varepsilon\,w^{1/2}$ $\varepsilon \geqq 1$

$$\frac{3\,v}{f\,(v - w) + (4\,vw)^{1/3}} = \frac{6\,\varepsilon}{f\,(2\,\varepsilon - w^{1/2}) + 2\,\varepsilon^{1/3}} \leqq \frac{6\,\varepsilon}{f\,(2\,\varepsilon - 2 + \sqrt{3}) + 2}$$

(9) follows from

$$f \geqq \frac{\{2\,\varepsilon - 2\,(2 - \sqrt{3})\}\,(2 + \sqrt{3})}{2\,\varepsilon - (2 - \sqrt{3})} \geqq 2 + \sqrt{3}.$$

4. Writing again $v = 2\,\varepsilon\,w^{1/2}$, we have

$$v^2 - vw + w^2 - v - w + 1 = (w - w^{1/2} + 1)^2 + 2\,(\varepsilon - 1)\,w^{1/2}\,(-w + 2w^{1/2} + 2\,\varepsilon\,w^{1/2} - 1)$$
$$\geqq (w - w^{1/2} + 1)^2.$$

As before (9) follows from

$$w - w^{1/2} + 1 \geqq \frac{6\,\varepsilon}{f\,(2\,\varepsilon - w^{1/2}) + 2},$$

which is equivalent to

$$(2 - w^{1/2})\ (f\,w - f\,w^{1/2} - 2\,w^{1/2} + f - 2) + 2\,(\varepsilon - 1)\,(f\,w - f\,w^{1/2} + f - 3) \geqq 0.$$

The last inequality is obvious since the second term is $\geqq 0$ already for $f > 4$ and the first term vanishes for the first time in the w interval for $f = 2 + 4.3^{-1/2}$. It is impossible to find a better constant in (9), since the combination $v = -2 + 2\sqrt{3}$, $w = 4 - 2\sqrt{3}$ for which $v^2 = 4w$ transforms (9) into an equality.

§ 7. The results of the last sections will be applied to the remainders R_n and U_n occurring in (2) and (3). Substituting (7) and (8) in (6) we obtain

$$|R_n(\zeta)| \leqq (2/\pi)^{2/3}(1 + h\zeta)\cdot(1/\pi)\int_1^\infty (t^2 - 1)^{-1/2}(\theta^2 + (\pi^2/4))^{-(n/6)}\, T\, dt$$
$$\text{if } \arg\zeta = 0;$$
$$|R_n(\zeta)| \leqq (2/\pi)^{2/3}\cdot(1/\pi)\int_1^\infty (t^2 - 1)^{-1/2}(\theta^2 + (\pi^2/4))^{-(n+2/6)}\, T\, dt$$
$$\text{if } \arg\zeta = (2\pi/3).$$

From § 3 it follows

$$(10)\quad 1/\pi\int_1^\infty (t^2 - 1)^{-1/2}(\theta^2 + (\pi^2/4)^{-(k/6)}\, dt \leqq (2/\pi)^{k/3+1}\left\{0\cdot 64 + \frac{3.2^{-k/6}}{k}\right\}$$

valid for all positive integer values of k.

Denoting the left hand side of (10) by e_k we get

$$|R_n(\zeta)| \leqq (2/\pi)^{2/3}(1 + h_1\zeta)(1 + h_2\zeta)\, e_{n-1} \quad (\arg\zeta = 0;\ n \equiv 4 \pmod 6),$$
$$|R_n(\zeta)| \leqq (2/\pi)^{2/3}(1 + h_1\zeta)\, e_{n-1} \quad (\arg\zeta = 0;\ n \equiv 0 \pmod 6);$$
$$|R_n(\zeta)| \leqq (2/\pi)^{2/3}(1 + h_2\zeta)\, e_{n+1} \quad (\arg\zeta = 2\pi/3;\ n \equiv 4 \pmod 6);$$
$$|R_n(\zeta)| \leqq (2/\pi)^{2/3}\, e_{n+1} \quad (\arg\zeta = 2\pi/3;\ n \equiv 0 \pmod 6),$$

where

$$h_1 = (2 + \sqrt{3})(2/\pi)^{1/3} = 3\cdot 2105 \qquad \text{and} \qquad h_2 = \sqrt{3}\,(2/\pi)^{1/3} = 1\cdot 4900.$$

Hence we find for U_n the upper bound

$$|U_n| \leqq (2^{7/6}\omega^{1/3}/\pi^{5/3})\left\{|\cos\omega\pi/2|\int_0^\infty e^{-\omega\zeta^3}(1 + h_1\zeta)(1 + h_2\zeta)\,\zeta^n\, d\zeta\cdot e_{n-1} + \right.$$
$$\left. + \int_0^\infty e^{-\omega\zeta^3}(1 + h_2\zeta)\,\zeta^n\, d\zeta\cdot e_{n+1}\right\}.$$

if $n \equiv 4 \pmod 6$ and

$$|U_n| \leqq (2^{7/6}\omega^{1/3}/\pi^{5/3})\left\{|\cos\omega\pi/2|\int_0^\infty e^{-\omega\zeta^3}(1 + h_1\zeta)\,\zeta^n\, d\zeta\cdot e_{n-1} + \right.$$
$$\left. + \int_0^\infty e^{-\omega\zeta^3}\,\zeta^n\, d\zeta\cdot e_{n+1}\right\},$$

if $n \equiv 0 \pmod 6$.

Or finally

$$(11a)\quad |U_n| \leqq \frac{2^{7/6}\omega^{1/3}}{3\pi^{5/3}}\left\{|\cos\omega\pi/2|(\gamma_0 + c_1\gamma_1 + c_2\gamma_2)\, e_{n-1} + (\gamma_0 + c_3\gamma_1)\, e_{n+1}\right\},$$

if $n \equiv 4 \pmod 6$, and

11b) $$|U_n| \leqq \frac{2^{7/6}\,\omega^{1/3}}{3\,\pi^{5/3}} \{|\cos \omega\pi/2|\,(\gamma_0 + c_4\gamma_1)\, e_{n-1} + \gamma_0\, e_{n+1}\},$$

if $n \equiv 0 \pmod 6$, where

$$\gamma_j = \Gamma\left(\frac{n+j+1}{3}\right) \omega^{-\frac{n+j+1}{3}} \qquad (j = 0, 1, 2),$$

$$c_1 = 2\,(1 + \sqrt{3})\,(2/\pi)^{1/3} = 4{\cdot}7005; \quad c_2 = (3 + 2\sqrt{3})\,(2/\pi)^{2/3} = 4{\cdot}7837;$$

$$c_3 = \sqrt{3}\,(2/\pi)^{2/3} = 1{\cdot}4900; \quad c_4 = (2 + \sqrt{3})\,(2/\pi)^{1/3} = 3{\cdot}2105.$$

To show the practible applicability of (4) and (11) a numerical example will be given:

$$M_{4,0}\,(8) = 1{\cdot}4494 - 0{\cdot}0136 - 0{\cdot}0013 + U_{10} = 1{\cdot}4345 + U_{10}.$$

(11a) gives

$$|U_{10}| \leqq 0{\cdot}0009$$

Further we have

$$m_6 = -\,0{\cdot}047 \text{ and } e_6 = 0{\cdot}184.$$

Koninklijke-Shell laboratorium,
Amsterdam