MATHEMATICS

SOME THEOREMS IN THE THEORY OF UNIFORM DISTRIBUTION

BY

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§ 1. In a preceding paper [1] we deduced the following

Theorem. *If $f(t)$ is a differentiable function, defined for $t \geqslant 0$, and if $f'(t)$ is bounded with*

$$t\, f'(t) \to 0 \text{ for } t \to \infty,$$

then $f(t)$ is not C_{I}-uniformly distributed (mod 1). [1] [2])

In the present paper we prove the following generalisations:

Theorem I. *If $f(t)$ is a differentiable function, defined for $t \geqslant 0$, and if $t\, f'(t)$ is bounded with*

$$t\, f'(t) \to A \text{ for } t \to \infty,$$

where A is a fixed number, then $f(t)$ is not C_{I} uniformly distributed (mod 1).

This Theorem is a special case of the following

Theorem II. *If $f(t)$ is a differentiable function, defined for $t \geqslant 0$, if $tf'(t)$ is bounded, and if there exists a fixed $T^* \geqslant 0$ such that for $t > T^*$*

$$|t\, f'(t) - A| < B < (1/2\pi),$$

where A and B are fixed numbers, then $f(t)$ is not C^{I} uniformly distributed (mod 1).

Proof of Theorem II.

We apply the C-test. We shall prove that

$$I = (1/T) \int_0^T e^{2\pi i h f(t)}\, dt$$

with $h = 1$ does not tend to zero for $T \to \infty$.

By integration by parts we have

$$I = e^{2\pi i f(T)} - (2\pi i/T) \int_0^T t\, f'(t)\, e^{2\pi i f(t)}\, dt.$$

[1]) For the definitions of C^{I}, C^{II} and C^{III} uniform distribution (mod 1) we refer to [2].

[2]) The condition, $f(t) \to \infty$ for $t \to \infty$, which occurs in Theorem 3 of [1], can be omitted.

Hence

$$(1+2\pi i A)\, I = e^{2\pi i f(T)} - (2\pi i/T)\int_0^T \{t\, f'(t) - A\}\, e^{2\pi i f(t)}\, dt$$

$$= e^{2\pi i f(T)} - (2\pi i/T)\int_0^{T^*} - (2\pi i/T)\int_{T^*}^{T}$$

$$= e^{2\pi i f(T)} - I_1 - I_2.$$

From this it follows:

$$\sqrt{1+4\pi^2 A^2}\, |I| \geqslant 1 - |I_1| - |I_2|. \tag{1}$$

For an arbitrarily chosen positive number ε there exists a $T^{**} \geqslant T^*$ such that for $T > T^{**}$

$$|I_1| < \varepsilon, \tag{2}$$

since $t\, f'(t)$ is bounded. For all $T > T^*$ we have furthermore

$$|I_2| < \frac{2\pi\,(T-T^*)}{T}\, B < 2\pi B. \tag{3}$$

From (1), (2) and (3) it follows that

$$\sqrt{1+4\pi^2 A^2}\, |I| > 1 - 2\pi B - \varepsilon \quad \text{for } T > T^{**}. \tag{4}$$

Since the positive number ε may be chosen arbitrarily small, it follows from (4) and $2\pi B < 1$ that $|I|$ does not tend to zero for $T \to \infty$. This completes the proof.

Examples. The following functions, of which the behaviour with regard to the C-uniform distribution (mod 1) could not yet ascertained by the Theorems in [1], satisfy the assumptions of Theorem II:

$$(a) \quad f(t) \equiv \log t + c\int_0^t (\sin u/u)\, du,$$

where c is a fixed number with $|c| < 1/2\pi$.

For $c = 0$ we meet again the function $\log t$.

$$(b) \quad f(t) = A \log t \sin(\log\log t),$$

where A is a constant with $|A| < 1/2\pi\sqrt{2}$.

Hence these functions are not C^{I}-uniformly distributed (mod 1).

§ 2. In [1] we proved the following

Theorem. *If $f(t)$ is a differentiable function, defined for $t \geqslant 0$, and if $f'(t) > 0$ and monotonically non-decreasing for $t \geqslant 0$, then $f(t)$ is C^{III}-uniformly distributed* (mod 1).

N. H. Kuiper (Delft, Netherlands) reported us, that, if $f'(t)$ tends to a constant $c \neq 0$ for $t \to \infty$, the condition of the monotony of $f'(t)$ is not necessary for $f(t)$ being C^{I} uniformly distributed (mod 1). In order to prove this statement he made use of a method similar to that developed in [3].

For the case that $f'(t) \to c \neq 0$ for $t \to \infty$ we shall prove the following

Theorem III. *If $f(t)$ is a differentiable function, defined for $t \geqslant 0$, and if*

$$f'(t) \to c \text{ for } t \to \infty, \tag{5}$$

where c is a fixed number $\neq 0$, then $f(t)$ is C^{III}-uniformly distributed (mod 1).

Proof. Without loss of generality we assume that c is positive.

It follows from (5) that for T sufficiently large $f(t)$ possesses an inverse function $t = F(u)$. Then from (5) it follows that

$$F'(u) \to (1/c) > 0 \text{ for } u \to \infty.$$

Hence, for $u > U^* = U^*(\varepsilon) = f(T^*)$ we have

$$|F'(u) - (1/c)| < (\varepsilon/c),$$

where ε is an arbitrarily small positive number. Now, applying the C-test, we have for every fixed h, integer and $\neq 0$, and $T > T^*$

$$I = (1/T)\int_0^T e^{2\pi i h f(t)}\,dt = (1/T)\int_0^{T^*} + (1/T)\int_{T^*}^{T} =$$

$$(1/T)\int_0^{T^*} e^{2\pi i h f(t)}\,dt + (1/T)\int_{f(T^*)}^{f(T)} e^{2\pi i h u}\,F'(u)\,du =$$

$$(1/T)\int_0^{T^*} e^{2\pi i h f(t)}\,dt + (1/T)\int_{f(T^*)}^{f(T)} \{F'(u) - (1/c)\}\,e^{2\pi i h u}\,du +$$

$$(1/cT)\int_{f(T^*)}^{f(T)} e^{2\pi i h u}\,du = I_1 + I_2 + I_3.$$

It is obvious that $I_1 \to 0$ for $T \to \infty$. Furthermore we have

$$|I_2| < \frac{\varepsilon}{cT}\{f(T) - f(T^*)\} < \frac{\varepsilon\, f(T)}{cT}.$$

It follows from (5) that

$$\frac{f(T)}{T} \to c \text{ for } T \to \infty.$$

Hence

$$I_2 \to 0 \text{ for } T \to \infty.$$

Finally we have

$$|I_3| \leqslant \frac{1}{\pi\,|h|\,cT} \text{ and so } I_3 \to 0 \text{ for } T \to \infty.$$

Thus, for $T \to \infty$, the expression I tends to zero.

Hence: $f(t)$ is C^{I}-uniformly distributed (mod 1). Furthermore it follows from the *Theorem:*

If $f(t)$ $(t \geqslant 0)$ *is a differentiable function with* $f(t)/t$ *bounded, if* $f'(t) \geqslant \lambda > 0$ *with fixed* λ, *and if* $f(t)$ *is* C^{I}*-uniformly distributed* (mod 1), *then* $f(t)$ *is also* C^{III}*-uniformly distributed* (mod 1),

proved in [4], that the function of Theorem III is also C^{III}-uniformly distributed (mod 1).

Example. The function

$$f(t) \equiv t + \frac{\sin t}{t}$$

satisfies the assumptions of Theorem III with $c = 1$, so that $f(t)$ is C^{III}-uniformly distributed (mod 1).

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LITERATURE

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