MATHEMATICS

PRIME-REPRESENTING FUNCTIONS

BY

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W. H. MILLS [1] proved the following

Theorem. There exists a real number A, such that $[A^{3^x}]$, where $[R]$ denotes the greatest integer less than or equal to R, is a prime number for every positive integer x.

In order to prove this statement he makes use of the inequality, deduced by A. E. INGHAM [2]

$$p_{n+1} - p_n < K\, p_n^{5/8},$$

where p_n denotes the n^{th} prime number and K is a positive integer.

We generalise the theorem mentioned above as follows:

Theorem. [1]) For each integer $c \geqq 3$ a real number A depending only on c can be found such that $[A^{c^x}]$ is a prime number for all positive integers x.

The proof runs as follows. The integers $a = (c-1)\ w - 1$ and $b = cw - 1$ satisfy the relations

$$(1) \qquad (a,\ b) = 1;\ a < b-1;\ a/b \geqq 5/8;\ ca + 1 \equiv 0 \pmod{b}.$$

Then

$$(2) \qquad p_{n+1} - p_n < K\, p^{a/b},$$

where K has the same value as before.

The lemma used by MILLS can be established here in the following form:

If a, b and c are positive integers satisfying (1) and (2), and if N is an integer with

$$(3) \qquad N > K^b,$$

then there exists a prime p such that

$$N^c < p < (N+1)^c - 1.$$

Proof. If p_n is the greatest prime less than N^c, then we have, using (1), (2) and (3)

$$p_n < N^c < p_{n+1} < p_n + K p_n^{a/b} < N^c + K\, N^{ca/b} < N^c + N^{(ca+1)/b} < (N+1)^c - 1.$$

1) We are indepted to Prof. J. G. VAN DER CORPUT for a generalisation of the Theorem as it was originally stated by us.

If now P_0 is a prime greater than K^b, then we are able to construct a sequence of primes P_0, $P_1, \ldots$ such that

$$(4) \qquad P_n^c < P_{n+1} < (P_n + 1)^c - 1 \qquad (n = 0, 1, 2, \ldots).$$

In the same way, as it has been done by MILLS for the case $a = 5$, $b = 8$, $c = 3$, we can prove by (4), that the sequence $P_n^{c^{-n}}$ is a bounded increasing sequence, and that therefore $P_n^{c^{-n}}$ tends to a limit, say A.

The boundedness of $P_n^{c^{-n}}$ follows from the inequality $P_n^{c^{-n}} < (P_n + 1)^{c^{-n}}$, where the sequence $(P_n + 1)^{c^{-n}}$ is a decreasing sequence, which also follows from (4).

So we have

$$P_n^{c^{-n}} < A < (P_n + 1)^{c^{-n}}$$

or

$$P_n < A^{c^n} < P_n + 1.$$

This means that $[A^{c^x}]$ $(x = 1, 2, \ldots)$ is a prime-representing function.

For two different values of c, say c and γ, one of the corresponding sequences $P_n(c)$ and $p_n(\gamma)$ cannot be a subsequence of the other. For instance, if we assume, that the sequence $p_n(\gamma)$ is a subsequence of $P_n(c)$, then there would be a subsequence $Q_n(c)$ of $P_n(c)$, such that $Q_n^{\gamma^{-n}}$ and $Q_n^{c^{-n}}$ would tend to the same limit, and this is impossible on account of $c \neq \gamma$.

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LITERATURE

1. MILLS, W. H., A prime representing function, Bull. Am. Math. Soc., **53**, 604.
2. INGHAM, A. E., On the difference between consecutive primes, Quart. J. Math. Oxford Ser., **8**, 255–266 (1937).