MATHEMATICS

ON THE UNIFORM DISTRIBUTION OF THE VALUES OF FUNCTIONS OF n VARIABLES

BY

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§ 1. *Introduction.*

In a previous paper KUIPERS and the author of this paper [1] introduced the notion of a (mod. 1) continuously- (or C-)uniformly distributed function $f(t)$ of one variable t. This notion is an analogue of that of a (mod. 1) discretely- (or D-) uniformly distributed sequence of numbers $f(n)$ $(n = 1, 2, \dots)$.

In the present paper I shall give an extension of the theory in [1], by considering instead of a function $f(t)$ of one variable now a function $f(t_1, \dots, t_n)$ of the variables $t_1, \dots, t_n$. For the D-case we recall the definition of D-uniform distribution (mod 1) of a system of m functions each depending on n variables given by WEIJL [2] and VAN DER CORPUT [3]. We mention this definition [4].

D-Definition. *Let m and n be given positive integers, and let F be a sequence of n-dimensional intervals*

$$Q:\quad a_\mu \leqq x_\mu < b_\mu \quad (a_\mu \text{ and } b_\mu \text{ integer}, \ \mu = 1, \dots, n),$$

where the number $N = N(Q)$ of lattice-points $(x) = (x_1, \dots, x_n)$ of Q tends to infinity if Q runs through the sequence F. In each lattice-point of Q a system of m real functions $f_\nu(x) = f_\nu(x_1, \dots, x_n)$ $(\nu = 1, \dots, m)$ is defined. This system of functions f_ν is said to be D-uniformly distributed (mod 1) in the intervals Q of F, if for each system of m numbers $\gamma_1, \dots, \gamma_m$ with $0 \leqq \gamma_\nu \leqq 1$ the number $N' = N'(Q) = N'(Q, \gamma_1, \dots, \gamma_m)$ of lattice-points (x) of Q with

$$0 \leqq f_\nu < \gamma_\nu \ (\text{mod } 1) \qquad (\nu = 1, \dots, m)$$

satisfies the relation:

$$\lim \frac{N'(Q)}{N(Q)} = \gamma_1 \gamma_2 \cdots \gamma_m,$$

if Q runs through the sequence F.

VAN DER CORPUT [3] proved the following theorem on uniform distribution.

D-test. *It is necessary and sufficient for the D-uniform distribution (mod 1) of the system of functions $f_\nu(x_1, \dots, x_n)$ $(\nu = 1, \dots, m)$ that,* *for any lattice-point* $(h_1, \ldots, h_m) \neq (0, \ldots, 0)$ *this system satisfies the relation:*

$$\lim \frac{1}{N(Q)} \sum_{(x) \text{ in } Q} e^{2\pi i \{h_1 f_1(x) + \ldots + h_m f_m(x)\}} = 0,$$

if Q *runs through the sequence* F.

In § 2 I shall give an analogous definition and an analogous theorem for the case of C-uniform distribution (mod 1). In §§ 3 and 4 I shall apply this C-test in the special case of one function of n variables. § 3 deals with a class of functions which are not C-uniformly distributed. The functions considered in § 4 are shown to be C-uniformly distributed. § 5 gives some examples.

§ 2. *C-definition and C-test.*

An n-dimensional extension of the definition of C-uniform distribution may be formulated as follows.

C-Definition. *Let* m *and* n *be given positive integers and let* F *be a sequence of n-dimensional intervals*

$$Q: \quad 0 \leqq S_\mu \leqq t_\mu < T_\mu \qquad (\mu = 1, \ldots, n),$$

where the measure of Q *tends to infinity, if* Q *runs through* F. *For all points* $(t) = (t_1, \ldots, t_n)$ *of all* Q *a system of n real measurable functions* $f_\nu(t_1, \ldots, t_n)$ $(\nu = 1, \ldots, m)$ *is defined.*

Let $\gamma_1, \ldots, \gamma_m$ *be* m *numbers with* $0 \leqq \gamma_\nu < 1$ $(\nu = 1, \ldots, m)$. *We denote by*

$$\theta_\nu\,(0, \gamma_\nu, f_\nu\,(t_1, \ldots, t_n)) \qquad (\nu = 1, \ldots, m)$$

the following characteristic functions:

$$\theta_\nu = 1 \text{ for } 0 \leqq f_\nu\,(t_1, \ldots, t_n) < \gamma_\nu \text{ (mod 1)},$$
$$\theta_\nu = 0 \text{ elsewhere.}$$

The measure of the set of points $(t_1, \ldots, t_n)$ *in* Q *satisfying:*

$$0 \leqq f_\nu(t_1, \ldots t_n) < \gamma_\nu \text{ (mod 1)} \qquad (\nu = 1, \ldots, m)$$

is the Lebesgue-integral:

$$I\,(Q) = \int_{S_1}^{T_1} \ldots \int_{S_n}^{T_n} \theta_1\,(0, \gamma_1, f_1) \ldots \theta\,(0, \gamma_m, f_m)\, dt_1 \ldots dt_n.$$

Now we define:

The system of functions $f_\nu(t_1, \ldots, t_n)$ *is said to be C-uniformly distributed (mod 1) in the intervals* Q *of* F, *if for any system of fixed* γ_ν *the system* f_ν *satisfies:*

$$\lim \frac{I\,(Q)}{(T_1 - S_1) \ldots (T_n - S_n)} = \gamma_1 \ldots \gamma_m,$$

if Q *runs through* F.

Analogous to the D-test we have now:

C-test. *It is necessary and sufficient for the C-uniform distribution (mod 1) of the system of functions $f_\nu(t_1, \ldots, t_n)$ $(\nu = 1, \ldots, m)$ in the intervals Q of F that, for any lattice-point $(h_1, \ldots, h_m) \neq (0, \ldots, 0)$ the system f_ν satisfies the relation:*

$$\lim \frac{1}{(T_1-S_1)\ldots(T_n-S_n)} \int_{S_1}^{T_1} \ldots \int_{S_n}^{T_n} e^{2\pi i \{h_1 f_1(t) + \ldots + h_m f_m(t)\}} dt_1 \ldots dt_n = 0,$$

if Q runs through F.

Remark. The proof of this test is quite analogous to that of the D-test given by VAN DER CORPUT [3]. For the case of one function with one variable, and a system of n functions of one variable KUIPERS gave the C-proof in his dissertation [5].

§ 3. *Functions which are not C-uniformly distributed.*

Theorem 1. *Let F be a sequence of intervals*

$$Q: \quad 0 \leqq S_\mu \leqq t_\mu < T_\mu \qquad (\mu = 1, \ldots, n),$$

where the measure of Q and $T_n - S_n$ tend to infinity, if Q runs through F. Let $f(t_1, \ldots, t_n)$ be a function, defined for all $(t) = (t_1, \ldots, t_n)$ of all Q, and let $f(t)$ possess for these points (t) first partial derivatives with respect to each variable. Furthermore we suppose that $f(t)$ has the following properties:

(1⁰)*) $\left| t_n \frac{\partial f}{\partial t_n} \right| < M$ *for all $t = (t_1, \ldots, t_n)$ in all Q, where M is a fixed positive number.*

$$(2^0) \qquad \lim_{t_n \to \infty} t_\nu \frac{\partial f}{\partial t_\nu} = 0 \qquad (\nu = 1, \ldots, n)$$

uniformly in $(t_1, \ldots, t_{n-1})$.

Then $f(t_1, \ldots, t_n)$ is not C-uniformly distributed (mod 1) in the intervals Q of F.

Proof. Putting

$$P_l = (T_1 - S_1) \ldots (T_l - S_l),$$

we have for each fixed integer $h \neq 0$:

$$I = \frac{1}{P_n} \int_{S_1}^{T_1} \ldots \int_{S_n}^{T_n} e^{2\pi i h f(t_1, \ldots, t_n)} dt_1 \ldots dt_n =$$

$$= \frac{1}{P_n} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} \Big\{ T_n e^{2\pi i h f(t_1, \ldots, t_{n-1}, T_n)} - S_n e^{2\pi i h f(t_1, \ldots, t_{n-1}, S_n)} +$$

$$- 2\pi i h \int_{S_n}^{T_n} t_n \frac{\partial f(t_1, \ldots, t_n)}{\partial t_n} e^{2\pi i h f(t_1, \ldots, t_n)} dt_n \Big\} dt_1 \ldots dt_{n-1}.$$

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*) *Note added while correcting proofsheets:*
This assumption (1⁰) can be omitted without considerable alternation in the proof.

From assumption (2^0) it follows that, given a positive number ε, there exists a T_n^* such that for $t_n \geqq T_n^* = T_n(\varepsilon)$ (independent of $t_1, \ldots, t_{n-1}$):

$$\left| t_n \frac{\partial f(t_1,\ldots,t_n)}{\partial t_n} \right| < \frac{\varepsilon}{4\pi |h|}. \tag{1}$$

Hence, if $T_n^* > S_n$, and S_n is bounded:

$$\left\{ \begin{aligned} &\left| \frac{2\pi h}{P_n} \int_{S_1}^{T_1} \ldots \int_{S_n}^{T_n} t_n \frac{\partial f(t_1,\ldots,t_n)}{\partial t_n} e^{2\pi i h f(t_1,\ldots,t_n)}\, dt_1 \ldots dt_n \right| \\ &\leqq \left| \frac{2\pi h}{P_n} \int_{S_1}^{T_1} \ldots \int_{S_n}^{T_n^*} \right| + \left| \frac{2\pi h}{P_n} \int_{S_1}^{T_1} \ldots \int_{T_n^*}^{T_n} \right| < \frac{2\pi |h| (T_n^* - S_n) M}{T_n - S_n} + \\ &+ \frac{T_n - T_n^*}{T_n - S_n} \frac{\varepsilon}{2} \end{aligned} \right. \tag{2}$$

(by assumption (1^0) and (1)), and so it is obvious that (2) tends to zero, if $T_n - S_n \to \infty$.

If S_n is not bounded, then for S_n large enough the expression (2) is $< \varepsilon/2$, so that also in this case (2) tends to zero if $T_n - S_n \to \infty$.

Now we consider

$$\frac{1}{P_n} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} \{T_n e^{2\pi i h f(t_1,\ldots,t_{n-1},T_n)} - S_n e^{2\pi i h f(t_1,\ldots,t_{n-1},S_n)}\}\, dt_1 \ldots dt_{n-1} =$$

$$= \frac{1}{P_n} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} \Big\{ (T_n - S_n)\, e^{2\pi i h f(t_1,\ldots,t_{n-1},T_n)} + S_n \{ e^{2\pi i h f(t_1,\ldots,t_{n-1},T_n)} +$$

$$- e^{2\pi i h f(t_1,\ldots,t_{n-1},S_n)} \} \Big\}\, dt_1 \ldots dt_{n-1} =$$

$$= \frac{1}{P_{n-1}} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} e^{2\pi i h f(t_1,\ldots,t_{n-1},T_n)}\, dt_1 \ldots dt_{n-1} + I_n.$$

If S_n is bounded, then

$$|I_n| \leqq \frac{2 S_n}{T_n - S_n}, \text{ and so } I_n \to 0 \text{ if } T_n - S_n \to \infty.$$

If S_n is not bounded, then, using the inequality:

$$| e^{2\pi i u} - e^{2\pi i v} | \leqq 2\pi\, | u - v |,$$

we have:

$$|I_n| \leqq \frac{1}{P_n} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} S_n 2\pi |h|\, | f(t_1,\ldots,t_{n-1},T_n) - f(t_1,\ldots,t_{n-1},S_n) |\, dt_1 \ldots dt_{n-1}$$

$$= \frac{2\pi |h|}{P_{n-1}} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} S_n \left| \frac{\partial f(t_1,\ldots,t_{n-1},\xi)}{\partial t_n} \right| dt_1 \ldots dt_{n-1}$$

(where $S_n < \xi < T_n$)

$$< \frac{2\pi\,|h|}{P_{n-1}} \int_{S_1}^{T_1} \cdots \int_{S_{n-1}}^{T_{n-1}} \xi \left| \frac{\partial f(t_1,\ldots,t_{n-1},\xi)}{\partial t_n} \right| dt_1 \ldots dt_{n-1}.$$

From assumption (2^0) it follows that this expression tends to zero, if $T_n - S_n \to \infty$, hence $I_n \to 0$ also in this case.

Now we apply the same argument to the integrals:

$$\frac{1}{P_{n-1}} \int_{S_1}^{T_1} \cdots \int_{S_{n-1}}^{T_{n-1}} e^{2\pi i h f(t_1,\ldots,t_{n-1},T_n)}\, dt_1 \ldots dt_{n-1},$$

$$\frac{1}{P_{n-2}} \int_{S_1}^{T_1} \cdots \int_{S_{n-2}}^{T_{n-2}} e^{2\pi i h f(t_1,\ldots,t_{n-2},T_{n-1},T_n)}\, dt_1 \ldots dt_{n-2},$$

and so on, so that I can be reduced to the expression

$$e^{2\pi i h f(T_1,\ldots,T_n)},$$

apart from a finite sum of terms, of which each tends to zero, if Q runs through F. Hence, by the C-test the function $f(t_1, \ldots, t_n)$ is not uniformly distributed (mod 1).

§ 4. *C-distributed functions.*

Theorem 2. *Let F be a sequence of intervals*

$$Q:\quad 0 \leqq S_\mu \leqq t_\mu < T_\mu \qquad (\mu = 1, \ldots, n),$$

where the measure of Q and $T_n - S_n$ tend to infinity if Q runs through F. Let $f(t) = f(t_1, \ldots, t_n)$ be a function defined for all $(t) = (t_1, \ldots, t_n)$ of all Q, and let $f(t)$ possess a partial derivative with respect to t_n with the properties:

$\frac{\partial f}{\partial t_n}$ *is monotonically non-decreasing in t_n for each fixed $(t_1, \ldots, t_{n-1})$;*

$\frac{\partial f}{\partial t_n} \geqq \lambda > 0$ *for λ fixed, independent of $t_1, \ldots, t_{n-1}$. Then $f(t_1, \ldots, t_n)$ is C-uniformly distributed (mod 1) in the intervals Q of F.*

Proof. Putting $P = (T_1 - S_1)\ldots(T_n - S_n)$ we have for every integer $h \neq 0$:

$$I = \frac{1}{P} \int_{S_1}^{T_1} \cdots \int_{S_n}^{T_n} e^{2\pi i h f(t_1,\ldots,t_n)}\, dt_1 \ldots dt_n$$

$$= \frac{1}{P} \int_{S_1}^{T_1} \cdots \int_{S_{n-1}}^{T_{n-1}} \left\{ \int_{f(t_1,\ldots,t_{n-1},S_n)}^{f(t_1,\ldots,t_{n-1},T_n)} \frac{e^{2\pi i h u}\, du}{\dfrac{\partial f(t_1,\ldots,t_n)}{\partial t_n}} \right\} dt_1 \ldots dt_{n-1}$$

$$= \frac{1}{P} \int_{S_1}^{T_1} \cdots \int_{S_{n-1}}^{T_{n-1}} \left\{ \frac{1}{\dfrac{\partial f(t_1,\ldots,t_{n-1},S_n)}{\partial t_n}} \int_{f(t_1,\ldots,t_{n-1},S_n)}^{f(t_1,\ldots,t_{n-1},\xi)} e^{2\pi i h u}\, du \right\} dt_1 \ldots dt_{n-1}$$

where $S_n < \xi < T_n$ (by the mean-value-theorem). Hence by our assumptions:

$$|I| \leqq \frac{1}{P} \int_{S_1}^{T_1} \int_{S_{n-1}}^{T_{n-1}} \frac{dt_1 \ldots dt_{n-1}}{\pi |h| \lambda} = \frac{1}{(T_n - S_n)\pi |h| \lambda},$$

and this expression tends to zero if $T_n - S_n \to \infty$. So $f(t_1, \ldots, t_n)$ is C-uniformly distributed (mod. 1).

Theorem 3. *Let F be a sequence of intervals*

$$Q: \quad 0 \leqq S_\mu \leqq t_\mu < T_\mu \qquad (\mu = 1, \ldots, n),$$

where the measure of Q and $T_n - S_n$ tend to infinity if Q runs through F.

Let $f(t) = f(t_1, \ldots, t_n)$ be a function, defined for all $(t) = (t_1, \ldots, t_n)$ of all Q, and let $f(t)$ possess a partial derivative with respect to t_n with the properties:

$\frac{\partial f}{\partial t_n} < 0$, *monotonically non-increasing, and continuous in t_n, uniformly in $t_1, \ldots, t_{n-1}$;*

$(T_n - S_n) \frac{\partial f(t_1, \ldots, t_{n-1}, T_n)}{\partial t_n} \to \infty$, *uniformly in $t_1, \ldots . t_{n-1}$, if $T_n - S_n \to \infty$.*

Then $f(t_1, \ldots, t_n)$ is C-uniformly distributed (mod 1) in the intervals Q of F.

Proof. Putting $P = (T_1 - S_1) \ldots (T_n - S_n)$ we have for every integer $h \neq 0$:

$$I = \frac{1}{P} \int_{S_1}^{T_1} \ldots \int_{S_n}^{T_n} e^{2\pi i h f(t_1, \ldots, t_n)} \, dt_1 \ldots dt_n$$

$$= \frac{1}{P} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} \left\{ \int_{f(t_1, \ldots, t_{n-1}, S_n)}^{f(t_1, \ldots, t_{n-1}, T_n)} \frac{e^{2\pi i h u} \, du}{\frac{\partial f(t_1, \ldots, t_n)}{\partial t_n}} \right\} dt_1 \ldots dt_{n-1}$$

$$= \frac{1}{P} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} \left\{ \frac{1}{\frac{\partial f(t_1, \ldots, t_{n-1}, T_n)}{\partial t_n}} \int_{f(t_1, \ldots, t_{n-1}\, \xi_1)}^{f(t_1, \ldots, t_{n-1}, T_n)} e^{2\pi i h u} \, du \right\} dt_1 \ldots dt_{n-1}$$

(where $S_n < \xi_1 < T_n$).

Hence

$$|I| \leqq \frac{1}{(T_1 - S_1) \ldots (T_{n-1} - S_{n-1}) \pi |h|} \int_{S_1}^{T_1} \ldots \int_{S_{n-1}}^{T_{n-1}} \frac{dt_1 \ldots dt_{n-1}}{(T_n - S_n) \frac{\partial f(t_1, \ldots, t_{n-1}, T_n)}{\partial t_n}}.$$

From our assumptions it follows that the last expression tends to zero if $T_n - S_n \to \infty$, so that also $I \to 0$. So $f(t_1, \ldots, t_n)$ is uniformly distributed (mod 1).

§ 5. *Examples.*

In the present paragraph we consider the sequence F of intervals

$$Q:\quad 2 < t_\mu < T \qquad (\mu = 1, \dots, n)$$

with $T \to \infty$.

It is easily seen that for the following functions $f(t) = f(t_1, \dots, t_n)$ the assumptions of Theorem 1 are satisfied, so that these functions are *not* C-uniformly distributed (mod 1) in the intervals Q of F:

$$f(t) = \lg \lg (t_1 \dots t_n);$$
$$f(t) = \lg \lg (t_1 + \dots + t_n);$$
$$f(t) = \sqrt[p]{\lg (t_1 \dots t_n)} \qquad (p > 1);$$
$$f(t) = \sqrt[p]{\lg (t_1 + \dots + t_n)} \qquad (p > 1).$$

From Theorem 2 it follows that the functions

$$f(t) = t_1 \dots t_n,$$
$$f(t) = t_1 + \dots + t_n,$$
$$f(t) = e^{t_1 \dots t_n},$$
$$f(t) = e^{t_1 + \dots + t_n}$$

are uniformly distributed (mod 1) in the intervals Q of F, and from Theorem 3 it follows that the functions

$$f(t) = \sqrt[p]{t_1 \dots t_n} \qquad (p > 1),$$
$$f(t) = \sqrt[p]{t_1 + \dots + t_n} \qquad (p > 1),$$
$$f(t) = \{\lg (t_1 \dots t_n)\}^p \qquad (p > 1),$$
$$f(t) = \{\lg (t_1 + \dots + t_n)\}^p \qquad (p > 1)$$

are C-uniformly distributed (mod 1) in the intervals Q of F.

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