CORRELATION PROBLEMS IN A ONE-DIMENSIONAL MODEL OF TURBULENCE. II *)

BY

J. M. BURGERS

(Mededeling no. 65b uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft)

(Communicated at the meeting of March 25, 1950)

11. *Functions occurring in the statistical description of the system.* — The object of the following sections is to obtain expressions for $\overline{v_1 v_2}$ and $\overline{v_1^2 v_2}$. For this purpose it will be necessary to introduce certain statistical functions connected with the system. At first sight one might suppose that there would exist an analogy between the problems to be considered here and some investigations by KAMPÉ DE FÉRIET on correlation coefficients associated with stationary random functions [4]). On closer inspection, however, it will be seen that the present system is far more complicated than the cases treated by KAMPÉ DE FÉRIET. The complication is due to the nature of the relations between the λ_i and τ_i. The values of these quantities are derived from two sets of increasing numbers, ξ_i for the λ_i, σ_i for the τ_i, in such a way that every λ_i is associated with both τ_{i-1} and τ_i; and every τ_i with both λ_i and λ_{i+1}. This makes it impossible to define simple independent elements out of which the system can be built up, and thus far I have not succeeded in finding a convenient and precise description of the total ensemble of states possible for the system, together with an indication of the way in which all particular cases out of this ensemble can be obtained and counted.

In view of this difficulty the following way of attack has been chosen. We assume that the system has been given at a certain instant (all deductions of this section and the next ones will refer to a single instant of time only; problems of development in the course of time will not come up before section 17). Since the system is of infinite extent, we can assume that in itself it will already contain all situations compatible with a given set of statistical features, so that it is not necessary to introduce a collection of systems, to be treated simultaneously. We can then, by measuring and counting, determine various statistical properties of this given system.

---

*) Continued from these Proceedings, p. 247—260.

[4]) See J. KAMPÉ DE FÉRIET, Les fonctions aléatoires stationnaires et la théorie statistique de la turbulence homogène, Ann. de la Soc. Scientifique de Bruxelles, 59, serie I, 145 (1939).

In the first place, starting from an arbitrary value of i, we make a list of the lengths of the consecutive segments:

$$\lambda_i,\ \lambda_{i+1},\ \lambda_{i+2}, \ldots, \lambda_{i+k}, \ldots$$

Such a list can be made beginning from any value of i.

If we give attention to the length of a *single* segment, we can determine a distribution function $f_1(\lambda)\,d\lambda$, defined in such a way that in a group of N single segments, chosen either consecutively or at random (without reference to their lengths), there will be $N\,f_1(\lambda)\,d\lambda$ segments with lengths between λ and $\lambda + d\lambda$, provided N is sufficiently large. The assumption concerning the statistical homogeneity of the system ensures that such a function $f_1(\lambda)$ will exist. Evidently we shall have:

$$\int_0^\infty f_1(\lambda)\,d\lambda = 1 \quad ; \quad \int_0^\infty f_1(\lambda)\,\lambda\,d\lambda = \overline{\lambda_i} = l \tag{36}$$

We shall further introduce distribution functions $f_k(\Lambda_k)\,d\Lambda_k$ for the total length

$$\Lambda_k = \lambda_{i+1} + \lambda_{i+2} + \ldots + \lambda_{i+k} = \xi_{i+k} - \xi_i \tag{37}$$

of *a set of k consecutive segments.* The function f_k is defined in such a way that in a series of N sets of k consecutive segments there will be $N\,f_k(\Lambda_k)\,d\Lambda_k$ sets with total lengths between Λ_k and $\Lambda_k + d\Lambda_k$. — When no ambiguity is to be feared we shall often write $f_k(\lambda)\,d\lambda$ instead of $f_k(\Lambda_k)\,d\Lambda_k$. The functions $f_k(\lambda)$ will satisfy the relations:

$$\int_0^\infty f_k(\lambda)\,d\lambda = 1 \quad ; \quad \int_0^\infty f_k(\lambda)\,\lambda\,d\lambda = \overline{\Lambda_k} = kl \tag{38}$$

We must expect that there will exist a connection between the functions f_k and f_1, which could be formulated mathematically if we should know the statistical relations existing between the lengths of consecutive segments. Without explicit calculation it can be observed that when the quantity

$$f_0 = \lim f_1(\lambda) \text{ for } \lambda \to 0$$

has a finite value different from zero, the function $f_k(\lambda)$ will be of order λ^{k-1} for $\lambda \to 0$. It must be kept in mind that f_0 is of the dimension: $(\text{length})^{-1}$.

It seems probable that all functions $f_k(\Lambda_k)$ will decrease to zero exponentially when Λ_k increases without limit. We shall introduce the assumption that

$$\sum_{k=1}^{\infty} f_k(\eta) \quad \text{converges for every finite value of } \eta \tag{39}$$

We can even assume that

$$\sum_{k=1}^{\infty} f_k(\eta) \cong 1/l \tag{39a}$$

provided η is sufficiently large. For $\Sigma f_k(\eta)\, d\eta$ is the probability to find a vertical segment at a point ξ_{i+h} (where h is not known a priori) satisfying the condition:

$$\xi_i + \eta < \xi_{i+h} < \xi_i + \eta + d\eta.$$

This probability must become independent of η when η is large enough.

The assumption (39) entails that

$$\sum_{k=1}^{\infty} \int_0^{\eta} f_k(\eta)\, \eta^m\, d\eta \tag{40}$$

will be convergent for every finite value of η, m being an arbitrary positive quantity (independent of k). With $m = 0$ we shall have

$$\sum_{k=1}^{\infty} \int_0^{\eta} f_k(\eta)\, d\eta = \frac{\eta}{l} - \text{constant} \tag{40a}$$

for large η, in those cases where (39a) holds.

A distribution function $F_k(\lambda_i, \Lambda_k)\, d\lambda_i\, d\Lambda_k$ for the simultaneous values of λ_i and $\Lambda_k = \lambda_{i+1} + \ldots + \lambda_{i+k}$ in a set of $(k+1)$ consecutive segments, will occur as an auxiliary function at a certain point of the deductions.

As far as the arrangement of the segments λ_i is concerned, the most important statistical features of the system are characterised by the functions f_k and F_k. These functions may differ from one type of system to another. For a given system they may also vary in the course of time.

12. Thus far no attention has been given to the values of the τ_i or ζ_i. We cannot simply associate a value τ_i with every λ_i, since the statistical connection between τ_i and λ_i cannot be different from that between τ_{i-1} and λ_i. To associate with every λ_i a half sum $\frac{1}{2}(\tau_{i-1} + \tau_i)$ would be possible in principle, but makes it difficult to find the values of the τ_i separately. (It will be attempted to make use of this idea in section 27.)

What we can do, however, for any given system, is to calculate mean values of quantities like τ_i, τ_i^2, $\tau_{i-1}\,\tau_i$ etc. *exclusively for segments* λ_i *of a given length* (more exactly: with a length between given limits λ and $\lambda + d\lambda$). Such "restricted" mean values (which, the same as ordinary mean values of the type indicated by $\overline{}$, are based on a process of counting), will be denoted by $\overline{\tau_i}^{*}$ etc. They are functions of the length λ assumed for the segments λ_i. It will be evident that $\overline{\tau_{i-1}}^{*} = \overline{\tau_i}^{*}$.

From the definition it follows that

$$\int_0^{\infty} f_1(\lambda)\, \overline{\tau_i}^{*}\, d\lambda = \overline{\tau_i} \tag{41}$$

Indeed, when both sides of the equation are multiplied by N, the right hand side gives the total of all τ_i for a group of N segments (either consecutive or chosen at random); on the left hand side the same sum appears split up in partial sums, each referring to a particular length λ and each having its proper number of terms $N f_1(\lambda)\, d\lambda$.

In a similar way we shall introduce restricted mean values referring to a fixed value of the length $\Lambda_k = \xi_{i+k} - \xi_i$. We shall be concerned in particular with $\overline{\tau_i \tau_{i+k}}^{*}$. Again we shall have the relation:

$$\int_0^\infty f_k(\Lambda_k)\, \overline{\tau_i \tau_{i+k}}^{*}\, d\Lambda_k = \overline{\tau_i \tau_{i+k}} \tag{42}$$

Restricted mean values of the type $\overline{\tau_i \tau_{i+k}}^{*}$ will form a further set of quantities which serve to characterise the statistical features of a given system.

There remains to consider the ζ_i. As has been mentioned in section **8**, when one ζ_i has been given all ζ_{i+k} can be calculated. The relation between these quantities follows from eq. (26) and can be put into the form:

$$\zeta_{i+k} = \zeta_i + \sum_{h=1}^{h=k} \lambda_{i+h} - \tfrac{1}{2}\tau_i - \tau_{i+1} - \ldots - \tau_{i+k-1} - \tfrac{1}{2}\tau_{i+k} \tag{43}$$

The assumption that mean values like $\overline{\zeta_i^2}$ can be defined, puts a certain restriction on the values of the λ_i and τ_i. Indeed, while the assumption (27) is sufficient to ensure that $\overline{\zeta_{i+k}} = 0$ when $\overline{\zeta_i} = 0$, there is a danger that even with a given value of $\overline{\zeta_i^2}$ over a certain limited domain of values of i, the mean value of $\overline{\zeta_{i+k}^2}$ for very large k might appear to be much larger, in consequence of uncorrelated fluctuations of the λ_i and τ_i. It is difficult to put the relevant condition into a concise mathematical form. It can be split up into two separate conditions, one referring to the λ_i, the other one to the τ_i, but it may be that in this way an unnecessary restriction of possibilities is introduced. The condition will turn up at a later point of the deductions in a somewhat disguised form (section **21**). Provisionally it is sufficient to work on the basis of the assumption that $\overline{\zeta_i^2}$ exists.

It might be imagined that the difficulty concerning the ζ_i could be evaded by assigning, to each segment λ_i, a definite hinge point. However, although the position of the hinge point assigned to a segment may be chosen arbitrarily with regard to the centre of that segment, in a series of consecutive segments $\lambda_i, \lambda_{i+1}, \lambda_{i+2}, \ldots$, the positions of the consecutive hinge points must satisfy the relation $\ldots \sigma_{i-1} < \sigma_i < \sigma_{i+1} < \sigma_{i+2} < \ldots$. This condition entails awkward relations to be fulfilled by the consecutive segments λ_i.

The calculations of $\overline{v_1 v_2}$ and $\overline{v_1^2 v_2}$, to be carried out in the following sections, will show in which way the various statistical quantities enter into the deductions.

We add that the invariant relations of section **8** (statistical homogeneity with respect to a shift in the starting point for i; and invariance with respect to a change of the direction in which i is counted, together with a change of sign of ζ_i) apply likewise to the restricted mean values.

13. *Calculation of* $\overline{v_1 v_2}$. — Provisionally we assume the jumps in the curve for $v(y)$ to be discontinuities. The correction to be introduced for the "rounding off" will be considered in section **15**. We write:

$$v_2 = v_1 + \beta\eta - \triangle,$$

where $\triangle$ is the amount to be subtracted in connection with the number of vertical segments between the points $y + \eta$ and y. We assume that y is situated in the segment λ_i, so that $\xi_{i-1} \leqslant y \leqslant \xi_i$. If $y + \eta$ is situated between ξ_{i+k} and ξ_{i+k+1}, the amount to be subtracted is $\triangle = \beta\,(\tau_i + {} + \tau_{i+1} + \ldots + \tau_{i+k})$ (compare fig. 6).

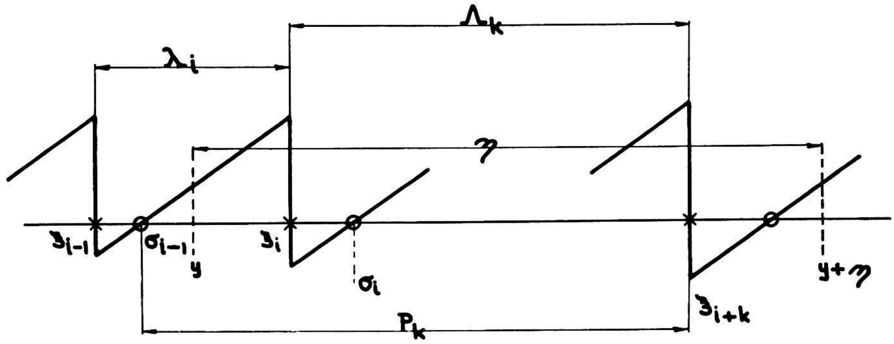

Fig. 6.

When we calculate

$$\int_{\xi_{i-1}}^{\xi_i} dy\, v_1\,(v_1 + \beta\eta - \triangle),$$

the first term is the same as occurs in the calculation of $\overline{v^2}$ and its mean value can be obtained from section **9**; the second one, apart from the constant factor $\beta\eta$, occurs in the calculation of $\overline{v}$ and consequently gives zero in the final result.

Coming to the amount $\triangle$, its first term $\beta\tau_i$ must be subtracted as soon as $y + \eta > \xi_i$. When $\lambda_i > \eta$, this will be the case for values of y within the range

$$\xi_i - \eta \leqslant y \leqslant \xi_i;$$

when $\lambda_i < \eta$, it will be the case for all values of y within the range

$$\xi_{i-1} \text{ (or, } \xi_i - \lambda_i) \leqslant y \leqslant \xi_i.$$

Giving attention to the values of v corresponding to the limiting values of y, the contribution to the integral is found to be:

$$-\tfrac{1}{2}\beta^2\tau_i\{(\zeta_i + \tfrac{1}{2}\tau_i)^2 - (\zeta_i + \tfrac{1}{2}\tau_i - \eta)^2\} \quad \text{for} \quad \lambda_i > \eta;$$
$$-\tfrac{1}{2}\beta^2\tau_i\{(\zeta_i + \tfrac{1}{2}\tau_i)^2 - (\zeta_i + \tfrac{1}{2}\tau_i - \lambda_i)^2\} \quad \text{for} \quad \lambda_i < \eta.$$

Since the probabilities or frequencies of the two cases depend on the distribution function $f_1(\lambda)\,d\lambda$ for the lengths λ_i, the mean contribution to the integral takes the form:

$$-\tfrac{1}{2}\beta^2\int\limits_{\eta}^{\infty} d\lambda\, f_1(\lambda)\,\overline{\tau_i\{(\zeta_i+\tfrac{1}{2}\tau_i)^2-(\zeta_i+\tfrac{1}{2}\tau_i-\eta)^2\}}^{\,*} -$$

$$-\tfrac{1}{2}\beta^2\int\limits_{0}^{\eta} d\lambda\, f_1(\lambda)\,\overline{\tau_i\{(\zeta_i+\tfrac{1}{2}\tau_i)^2-(\zeta_i+\tfrac{1}{2}\tau_i-\lambda)^2\}}^{\,*},$$

where use has been made of the notation introduced for restricted mean values referring to a particular value of λ_i. The expression can be transformed into:

$$-\tfrac{1}{2}\beta^2\int\limits_{0}^{\infty} d\lambda\, f_1(\lambda)\,\overline{\tau_i\{(\zeta_i+\tfrac{1}{2}\tau_i)^2-(\zeta_i+\tfrac{1}{2}\tau_i-\eta)^2\}}^{\,*} -$$

$$-\tfrac{1}{2}\beta^2\int\limits_{0}^{\eta} d\lambda\, f_1(\lambda)\,\overline{\tau_i\{(\zeta_i+\tfrac{1}{2}\tau_i-\eta)^2-(\zeta_i+\tfrac{1}{2}\tau_i-\lambda)^2\}}^{\,*}.$$

We consider the first integral in particular. According to the general relation exemplified in (41) it is equal to:

$$-\tfrac{1}{2}\beta^2\,\overline{\tau_i\{(\zeta_i+\tfrac{1}{2}\tau_i)^2-(\zeta_i+\tfrac{1}{2}\tau_i-\eta)^2\}},$$

which after some simple reductions yields the result:

$$\text{(I)}\qquad -\tfrac{1}{2}\beta^2(\overline{\tau_i^2}\,\eta-\overline{\tau_i}\,\eta^2) = -\tfrac{1}{2}\beta^2 l^3\left\{(1+\omega)\frac{\eta}{l}-\frac{\eta^2}{l^2}\right\}$$

It is not necessary to calculate the second integral, since that quantity will be included automatically in an expression to be considered presently.

14. *Continuation.* — We turn to the consideration of the term $\beta\tau_{i+k}$, which must be subtracted as soon as $y+\eta>\xi_{i+k}$. Since $y\leqslant\xi_i$, this can only occur provided $\Lambda_k<\eta$. Again we distinguish between two cases: if $\lambda_i+\Lambda_k>\eta$, the subtraction is to be made only for values of y within the range $\xi_{i+k}-\eta<y<\xi_i$; if $\lambda_i+\Lambda_k<\eta$, so that $\xi_{i+k}-\eta<$ $<\xi_{i-1}$, subtraction must be made for all values of y within the range $\xi_{i-1}<y<\xi_i$.

We introduce the notations:

$$\text{(44)}\qquad T_k=\tfrac{1}{2}\tau_i+\tau_{i+1}+\tau_{i+2}+\ldots+\tau_{i+k-1}+\tfrac{1}{2}\tau_{i+k}$$

$$\text{(44a)}\qquad P_k=\xi_{i+k}-\sigma_{i-1}=T_k+\tfrac{1}{2}\tau_i+\zeta_{i+k}$$

The values of v at the limits of the integration interval can then be written:

in the first case ...... $\beta(P_k-\eta)$, $\beta(P_k-\Lambda_k)$;

in the second case .... $\beta(P_k-\Lambda_k-\lambda_i)$, $\beta(P_k-\Lambda_k)$.

We now need a distribution function $F_k(\lambda_i, \Lambda_k)\, d\lambda_i\, d\Lambda_k$ for the simultaneous values of λ_i and Λ_k. The terms dependent on τ_{i+k} can appear only if $\Lambda_k < \eta$; the limits to be used depend on the difference between λ and $\eta - \Lambda_k$. Hence the contribution to the integral, after a slight re-arrangement, can be written:

$$
\text{(II)} \quad \left\{ \begin{array}{l} -\frac{1}{2}\beta^2 \int\limits_0^\eta d\Lambda_k \int\limits_0^\infty d\lambda_i\, F_k(\lambda_i, \Lambda_k)\, \overline{\tau_{i+k}\{(P_k-\Lambda_k)^2-(P_k-\eta)^2\}}^* - \\ -\frac{1}{2}\beta^2 \int\limits_0^\eta d\Lambda_k \int\limits_0^{\eta-\Lambda_k} d\lambda_i\, F_k(\lambda_i, \Lambda_k)\, \overline{\tau_{i+k}\{(P_k-\eta)^2-(P_k-\Lambda_k-\lambda_i)^2\}}^* \end{array} \right.
$$

where the restricted mean values now refer to fixed values of both λ_i and Λ_k. Giving attention to the definition of the function $F_k(\lambda_i, \Lambda_k)$ we see that its integral with respect to λ_i from 0 to ∞ yields the function $f_k(\Lambda_k)$. The first integral occurring in (II) conseqently transforms into:

$$
\text{(III)} \qquad -\frac{1}{2}\beta^2 \int\limits_0^\eta d\Lambda_k\, f_k(\Lambda_k)\, \overline{\tau_{i+k}\{(P_k-\Lambda_k)^2-(P_k-\eta)^2\}}^*
$$

where the restricted mean value simply refers to a fixed value of Λ_k, without reference to a particular value of λ_i.

In consequence of (40) the series obtained by summing (III) with respect to k will be convergent for every finite value of η. It follows that we may treat both parts of the expression (II) separately.

To reduce the second integral to a more convenient form we decrease k by one unit. At the same time we replace i by $i+1$, which can be done without affecting the mean value. By this process P_k (which first became P_{k-1}) is changed into $P_k - \tau_i$. Further Λ_k is changed into $\Lambda_{k-1} = \lambda_{i+2} + \ldots + \lambda_{i+k}$; and $\Lambda_k + \lambda_i$ becomes $\Lambda_{k-1} + \lambda_{i+1} = \lambda_{i+1} + \lambda_{i+2} + \ldots + \lambda_{i+k} = \Lambda_k$. The expression to be calculated consequently takes the form:

$$
-\frac{1}{2}\beta^2 \int\limits_0^\eta d\Lambda_{k-1} \int\limits_0^{\eta-\Lambda_{k-1}} d\lambda_{i+1}\, F_{k-1}(\lambda_{i+1}, \Lambda_{k-1})\, \overline{\tau_{i+k}\{(P_k-\tau_i-\eta)^2-(P_k-\tau_i-\Lambda_k)^2\}}^* .
$$

It will be recognised that in this integral we combine all cases in which $\Lambda_{k-1} + \lambda_{i+1} = \Lambda_k$ takes values from 0 to η, with the proper probability function. Hence the integral can be transformed into:

$$
\text{(IV)} \qquad -\frac{1}{2}\beta^2 \int\limits_0^\eta d\Lambda_k\, f_k(\Lambda_k)\, \overline{\tau_{i+k}\{(P_k-\tau_i-\eta)^2-(P_k-\tau_i-\Lambda_k)^2\}}^*
$$

We now combine the integrals (III) and (IV). When the expressions between the { } are worked out, the result of the combination is:

$$
\text{(V)} \qquad +\beta^2 \int\limits_0^\eta d\Lambda_k\, f_k(\Lambda_k)\, (\Lambda_k - \eta)\, \overline{\tau_i \tau_{i+k}}^*
$$

With $k = 1$ the integral (IV) reduces to

$$-\tfrac{1}{2}\beta^2 \int_0^{\eta} d\lambda_{i+1}\, f_1(\lambda_{i+1})\, \overline{\tau_{i+1}\{(\zeta_{i+1} + \tfrac{1}{2}\tau_{i+1} - \eta)^2 - (\zeta_{i+1} + \tfrac{1}{2}\tau_{i+1} - \lambda_{i+1})^2\}}^{\,*},$$

which, by changing $i + 1$ into i, will be seen to be precisely the quantity we had left aside at the end of the preceding section, when we considered the contributions derived from $\beta\tau_i$. Hence, when the expression (IV) is summed with respect to k from $k = 1$ onward, that quantity is automatically included. The convergence of the sum again follows from (40).

We introduce the functions:

$$\varphi_k(\eta) = l^{-2} \int_0^{\eta} d\lambda\, f_k(\lambda)\, \overline{\tau_i \tau_{i+k}}^{\,*} \tag{45}$$

$$\chi_k(\eta) = l^{-3} \int_0^{\eta} d\lambda\, f_k(\lambda)\, \lambda\, \overline{\tau_i \tau_{i+k}}^{\,*} \tag{46}$$

The final expression for $\overline{v_1 v_2}$ is obtained by combining the expression (I) of section **11** with (V) as found above. We must divide by l to pass from $\overline{\quad}$ mean values to mean values per unit length of the y-axis. Making use of (45) and (46) this gives:

$$\overline{v_1 v_2} = \overline{v^2} - \beta^2 l^2 \left\{ \frac{1+\omega}{2} \frac{\eta}{l} - \frac{\eta^2}{2l^2} + \sum_1^{\infty} \left(\frac{\eta}{l}\varphi_k - \chi_k \right) \right\} \tag{47}$$

The value of $\overline{v^2}$ must be taken from (30). — We mention the derivative:

$$\frac{\partial}{\partial \eta} \overline{v_1 v_2} = -\beta^2 l \left\{ \frac{1+\omega}{2} - \frac{\eta}{l} + \sum_1^{\infty} \varphi_k \right\} \tag{48}$$

(the derivatives of φ_k and χ_k drop out).

Since $f_k(\lambda)$ is of order λ^{k-1} for small λ, it follows that, for small η, the function φ_k is of order η^k and the function χ_k of order η^{k+1}. Hence the sum in (47) begins with a term in η^2 and that in (48) with a term in η.

The expressions (47) and (48) must satisfy the condition that $\overline{v_1 v_2}$ and $\partial(\overline{v_1 v_2})/\partial\eta$ shall vanish for infinite values of η. In order to prove this we must investigate the sums $\Sigma\varphi_k$ and $\Sigma\chi_k$. We will come back to this point in section **20**.

15. *Correction for the rounding off of the jumps in the function* $v(y)$. — The expression (47) is not valid for values of η of the order $\nu/(v_- - v_+)$, or in the present notation, $\nu/\beta\tau_i$. It cannot be continued analytically down to $\eta = 0$ and through $\eta = 0$ to negative values. Indeed (47) is not an even function of η; when it should be used both for positive and negative values, η must be replaced by its absolute value $|\eta|$.

A correction to (47), leading to a formula valid for very small values (positive or negative) of η, can be found without difficulty if we may assume that the distance between two consecutive jumps in the great majority of cases is large compared with $\nu/\beta\tau_i$. The frequency of cases where this is otherwise will be determined by $f_0\nu/\beta\tau_i$; since f_0 can be expected to be of order $1/l$, this frequency will be of the order $\nu/\beta l^2 = \nu t/l^2$ [5]). We assume that this quantity is small compared with unity.

The correction to be applied to (47) can be found by making use of the hyperbolic tangent function indicated in (24). For simplicity we take the origin for y at the point ξ_i; in the neighbourhood of this point we write:

$$v = \beta\,(y - \zeta_i) - \tfrac{1}{2}\,\beta\,\tau_i \tanh\,(\beta\,\tau_i\,y/4\nu),$$

where ζ_i is a constant. We calculate the integral of v_1v_2 (in which η is supposed to be small) over the interval $-L < y < +L$, L being a length of order l, but less than both λ_i and λ_{i+1}. The difference between the value of the integral for finite ν and its value for $\nu = 0$, is the correction to be applied in connection with the jump at ξ_i. We pass over the details of the calculation; when we may assume that $\beta\tau_i(L - |\eta|)/\nu = \tau_i(L - |\eta|)/\nu t$ is sufficiently large (say greater than 10), the result is:

$$\tfrac{1}{2}\,\beta^2\,\eta\,\tau_i^2\,(1 - \mathrm{ctnh}\,\beta\,\tau_i\,\eta/4\,\nu) \tag{49}$$

The correction to formula (47) is the mean value of this expression:

$$\frac{\beta^2\,\eta}{2\,l}\,\overline{\tau_i^2\,(1 - \mathrm{ctnh}\,\beta\,\tau_i\,\eta/4\,\nu)} \tag{49a}$$

For values of η small in comparison with $4\nu t/l$ we may develop the ctnh function, which gives:

$$\frac{\beta^2\,\eta}{2\,l}\,\overline{\tau_i^2} - \frac{2\,\beta\,\nu}{l}\,\overline{\tau_i} - \frac{\beta^3\,\eta^2}{24\,l\,\nu}\,\overline{\tau_i^3}\ldots,$$

or, making use of eqs. (27) and (31) and re-arranging:

$$-2\,\beta\,\nu + \beta^2\,l^2\,\frac{1+\omega}{2}\,\frac{\eta}{l} - \frac{\beta^3\,l^4}{\nu}\,\frac{1+\omega^*}{24}\,\frac{\eta^2}{l^2}\ldots \tag{49b}$$

There is thus a small correction to the value of $\overline{v^2}$ given in (30); while if (49b) is added to (47) the following expression is obtained for $\overline{v_1v_2}$, valid for $\eta \ll 4\nu t/l$:

$$\overline{v_1\,v_2} = \overline{(v^2)}_{\text{uncorrected}} - 2\,\beta\,\nu - \frac{\beta^3\,l^4}{\nu}\,\frac{1+\omega^*}{24}\,\frac{{}^2}{l^2} \tag{50}$$

The term with η^2 in (47) and the terms depending on the functions φ_k and χ_k have been omitted, since, for the values of η considered here, they are insignificant in comparison with the last term of (50).

It is now possible to apply eqs. (8) and (9). This gives the result:

$$\varepsilon = \tfrac{1}{12}\,\beta^3\,l^2\,(1 + \omega^*),$$

which is in conformity with (33).

---

[5]) Very small values of τ_i will be an exception, since the τ can only increase by coalescence. Hence for estimating orders of magnitude we may replace τ_i by the mean value l.

16. *Calculation of* $\overline{v_1^2 v_2}$. — We follow the same procedure as applied in sections **13** and **14** and calculate

$$\int_{\xi_{i-1}}^{\xi_i} dy\, v_1^2\,(v_1 + \beta\eta - \triangle).$$

The first term leads to the mean value of v^3, which is zero. The second term leads to

$$\text{(VI)} \qquad \beta\eta\,\overline{v^2} = \beta^3 l^3 \{\tilde{\omega} + \tfrac{1}{12}(1+\omega^*)\}\,\frac{\eta}{l}$$

[compare (30) and (31)].

The first term in $\triangle$, *viz.* $\beta\tau_i$, gives rise to two integrals of a type similar to those occurring in section **13**. The first one of these has the form

$$-\tfrac{1}{3}\beta^3 \int_0^\infty d\lambda\, f_1(\lambda)\, \overline{\tau_i\{(\zeta_i + \tfrac{1}{2}\tau_i)^3 - (\zeta_i + \tfrac{1}{2}\tau_i - \eta)^3\}}^{\,*},$$

which according to (41) is equal to:

$$-\tfrac{1}{3}\beta^3\, \overline{\tau_i\{(\zeta_i + \tfrac{1}{2}\tau_i)^3 - (\zeta_i + \tfrac{1}{2}\tau_i - \eta)^3\}}.$$

After a few reductions this yields the result:

$$\text{(VII)} \quad \begin{cases} -\beta^3\{(\overline{\tau_i\zeta_i^2} + \tfrac{1}{4}\overline{\tau_i^3})\,\eta - \tfrac{1}{2}\overline{\tau_i^2}\,\eta^2 + \tfrac{1}{3}\overline{\tau_i}\,\eta^3\} = \\ \qquad = \beta^3 l^4 \left\{-\left(\tilde{\omega} + \dfrac{1+\omega^*}{4}\right)\dfrac{\eta}{l} + \dfrac{1+\omega}{2}\dfrac{\eta^2}{l^2} - \dfrac{1}{3}\dfrac{\eta^3}{l^3}\right\} \end{cases}$$

The contribution depending on $\beta\tau_{i+k}$ can be transformed in a similar way as was followed in section **14**. We pass over the details; the result is

$$\text{(VIII)} \quad \begin{cases} -\tfrac{1}{3}\beta^3 \displaystyle\int_0^\eta dA_k\, f_k(A_k)\, \overline{\tau_{i+k}\{(P_k - A_k)^3 - (P_k - \eta)^3\}}^{\,*} - \\ \qquad -\tfrac{1}{3}\beta^3 \displaystyle\int_0^\eta dA_k\, f_k(A_k)\, \overline{\tau_{i+k}\{(P_k - \tau_i - \eta)^3 - (P_k - \tau_i - A_k)^3\}}^{\,*} = \\ = \beta^3 \displaystyle\int_0^\eta dA_k\, f_k(A_k)\, \{2\,(A_k - \eta)\, \overline{\tau_i\tau_{i+k}\,(P_k - \tfrac{1}{2}\tau_i)}^{\,*} - (A_k^2 - \eta^2)\,\overline{\tau_i\tau_{i+k}}^{\,*}\} \end{cases}$$

The first two expressions correspond to the integrals (III) and (IV), respectively, of section **14**; the final expression corresponds to (V).

According to (44*a*) $P_k - \frac{1}{2}\tau_i = T_k + \zeta_{i+k}$. Making use of the second invariant relation, we can replace ζ_{i+k} by $-\zeta_i$, which will not affect the mean value. It is thus possible to replace $2\,(P_k - \frac{1}{2}\tau_i)$ by $2T_k + \zeta_{i+k} - \zeta_i = T_k + A_k$ [compare (43)]. In this way (VIII) is transformed into

$$\text{(IX)} \qquad \beta^3 \int_0^\eta dA_k\, f_k(A_k)\, \{(A_k - \eta)\, \overline{T_k\tau_i\tau_{i+k}}^{\,*} - (A_k\eta - \eta^2)\, \overline{\tau_i\tau_{i+k}}^{\,*}\}$$

We put:

$$\Phi_k(\eta) = l^{-3} \int_0^\eta d\lambda\, f_k(\lambda)\, \overline{T_k \tau_i \tau_{i+k}}^* \tag{51}$$

$$X_k(\eta) = l^{-4} \int_0^\eta d\lambda\, f_k(\lambda)\, \lambda\, \overline{T_k \tau_i \tau_{i+k}}^* \tag{52}$$

Combining the expression (VI) with (VII) and (IX) divided by l, we arrive at the following formula for $\overline{v_1^2 v_2}$:

$$\left\{ \begin{aligned} \overline{v_1^2 v_2} = -\beta^3 l^3 \Big[& \frac{1+\omega^*}{6} \frac{\eta}{l} - \frac{1+\omega}{2} \frac{\eta^2}{l^2} + \frac{1}{3} \frac{\eta^3}{l^3} - \\ & - \sum_1^\infty \Big\{ \frac{\eta^2}{l^2} \varphi_k - \frac{\eta}{l} (\Phi_k + \chi_k) + X_k \Big\} \Big] \end{aligned} \right. \tag{53}$$

From (53) we derive:

$$\frac{\partial}{\partial \eta} (\overline{v_1^2 v_2}) = -\beta^3 l^2 \Big[\frac{1+\omega^*}{6} - (1+\omega) \frac{\eta}{l} + \frac{\eta^2}{l^2} - \sum_1^\infty \Big\{ \frac{2\eta}{l} \varphi_k - \Phi_k - \chi_k \Big\} \Big] \tag{54}$$

For small η (small compared with l, but not so small as $4\nu t/l$) the function Φ_k is of order η^k and the function X_k of order η^{k+1}. It follows that the expression between $\{\}$ in (53) begins with a term in η^2 and in (54) with a term in η. Formulae (53) and (54) are not valid down to $\eta = 0$ and cannot be continued analytically to negative values of η.

17. *Application of the fundamental equation* (12). — Now that we have obtained expressions for $\overline{v_1 v_2}$ and $\overline{v_1^2 v_2}$ it is possible to make use of equation (12) to investigate the change in time of some of the statistical quantities we have encountered.

As was mentioned in section 2, equation (12), when applied to $\eta = 0$, brings us back to eq. (4). With E as given by (32), ε by (33) and $\beta = 1/t$, we obtain (after multiplication by 2):

$$\frac{d}{dt} \Big[\frac{l^2}{t^2} \Big(\tilde{\omega} + \frac{1+\omega^*}{12} \Big) \Big] = -\frac{l^2}{t^3} \frac{1+\omega^*}{6} \tag{55}$$

When $\eta \gg 4\nu t/l$, we can use (12*a*) instead of eq. (12) and apply expressions (47) and (53) for $\overline{v_1 v_2}$ and $\overline{v_1^2 v_2}$ respectively. This gives

$$\left\{ \begin{aligned} & \frac{\partial}{\partial t} \Big[\frac{l^2}{t^2} \Big(\tilde{\omega} + \frac{1+\omega^*}{12} \Big) - \frac{l\eta}{t^2} \frac{1+\omega}{2} + \ldots \Big] = \\ & \qquad = -\frac{l^2}{t^3} \frac{1+\omega^*}{6} + \frac{l\eta}{t^3} (1+\omega) - \frac{l^2 \Phi_1}{t^3} + \ldots \end{aligned} \right. \tag{56}$$

where terms not written out are at least of order η^2. Comparison of the terms of zero order brings us back to eq. (55), which thus also follows from expression (53) for $\overline{v_1^2 v_2}$. Comparison of the first order terms gives a new relation:

$$\frac{d}{dt} \Big[\frac{l}{t^2} \frac{1+\omega}{2} \Big] = -\frac{l}{t^3} (1+\omega) + \frac{l^2}{t^3} \lim_{\eta \to 0} \frac{\Phi_1}{\eta} \tag{57}$$

Having regard to (51) and noting that $T_1 = \frac{1}{2}(\tau_i + \tau_{i+1})$, we find:

$$\lim_{\eta \to 0} (\Phi_1/\eta) = (f_0/2\,l^3)\,\overline{(\tau_i + \tau_{i+1})\,\tau_i\,\tau_{i+1}}^{\,*} \tag{58}$$

the restricted mean value referring to $\lambda_i \to 0$.

By eliminating t^{-2} from the left hand members we can simplify eqs. (55), (57) to:

$$\frac{d}{dt}\left[l^2\left(\tilde{\omega} + \frac{1+\omega^*}{12}\right)\right] = \frac{2\,l^2\,\tilde{\omega}}{t} \tag{59}$$

$$\frac{d}{dt}\,[l\,(1+\omega)] = \frac{f_0}{l\,t}\,\overline{(\tau_i + \tau_{i+1})\,\tau_i\,\tau_{i+1}}^{\,*} \tag{60}$$

18. *Direct derivation of eqs.* (59), (60). — It is of interest to deduce these equations in a different way, which will throw some light on the meaning of the terms occurring in them.

We begin with eq. (60). According to (31) the expression between brackets on the left hand side is equal to $\overline{\tau_i^2}/l$. If we multiply by a large length S which is independent of the time, we obtain $N\,\overline{\tau_i^2}$ (with $N = S/l$). This is equal to the sum of all τ_i^2 to be found in the length S. Now in consequence of the laws of motion, section 7 (V), the τ_i do not change in the course of time, unless two consecutive τ's coalesce. When this happens to τ_i and τ_{i+1}, the sum of the squares increases by the amount:

$$(\tau_i + \tau_{i+1})^2 - \tau_i^2 - \tau_{i+1}^2 = 2\,\tau_i\,\tau_{i+1}.$$

In order to find the frequency of this process we must determine the number of segments λ_{i+1} which decrease to zero in the element of time dt. Since for small λ we have $d\lambda_{i+1}/dt = -(\tau_i + \tau_{i+1})/2t$, the magnitude of λ_{i+1} may be at most $dt\cdot(\tau_i + \tau_{i+1})/2t$ and the number of such segments in the length S will be given by:

$$N\,f_0\,dt\,(\tau_i + \tau_{i+1})/2\,t \tag{61}$$

Hence the rate of increase of the sum of all τ_i^2 in S is given by the mean value $N\,f_0\,\overline{(\tau_i + \tau_{i+1})\,\tau_i\,\tau_{i+1}}^{\,*}/t$ (restricted mean value for $\lambda_i \to 0$), and we obtain:

$$d\,(N\,\overline{\tau_i^2})/dt = N\,f_0\,\overline{(\tau_i + \tau_i)\,\tau_i\,\tau_{i+1}}^{\,*}/t.$$

Division by the constant length S brings us back to (60).

To prove (59) we transcribe it in the same way and consider:

$$d\,[N\,(\overline{\tau_i\,\zeta_i^2} + \tfrac{1}{12}\,\overline{\tau_i^3})]/dt = 2\,N\,\overline{\tau_i\,\zeta_i^2}/t.$$

From section 7 (III), we have $d\zeta_i/dt = \zeta_i/t$. Hence:

$$d(\tau_i\,\zeta_i^2)/dt = 2\,\tau_i\,\zeta_i^2/t.$$

This already gives us the term on the right hand side.

Further, when τ_i and τ_{i+1} coalesce, τ_i and τ_{i+1} simply add up; while ζ_i and ζ_{i+1} vanish, being replaced by a single quantity equal to:

$$\zeta^0 = \zeta_i - \tfrac{1}{2}\tau_{i+1} = \zeta_{i+1} + \tfrac{1}{2}\tau_i$$

(compare the end of section **7**). It follows that the sum increases by an amount which can be written

$$\begin{aligned}\tau_i(\zeta_i - \tfrac{1}{2}\tau_{i+1})^2 + \tau_{i+1}(\zeta_{i+1} + \tfrac{1}{2}\tau_i)^2 + \tfrac{1}{12}(\tau_i + \tau_{i+1})^3 - \\ - \tau_i\zeta_i^2 - \tau_{i+1}\zeta_{i+1}^2 - \tfrac{1}{12}\tau_i^3 - \tfrac{1}{12}\tau_{i+1}^3 = \\ = \tau_i\tau_{i+1}(\tfrac{1}{2}\tau_i + \tfrac{1}{2}\tau_{i+1} + \zeta_{i+1} - \zeta_i) = 0,\end{aligned}$$

since the expression between ( ) vanishes at the moment of coalescence. Hence the coalescence of segments is without influence on the value of the sum, so that the relation we were considering is proved.

The same method can be applied to other quantities.

In the first place we refer to LOITSIANSKY's invariant J_0 for which an expression was obtained in section **10**. We consider formula (35) for J_0 and note that according to the laws of motion $\beta\zeta_i = \zeta_i/t$ is constant so long as coalescence does not take place. Hence the regular development of the system will not influence the value of J_0. To prove that J_0 neither changes through the effect of coalescence, we take J_0 in the form:

$$\tfrac{1}{4}\beta^2 \overline{\{(\zeta_i + \tfrac{1}{2}\tau_i)^2 - (\zeta_{i-1} - \tfrac{1}{2}\tau_{i-1})^2\}\, \Sigma\, \tau_{i+k}\zeta_{i+k}}.$$

It is not difficult to show that the sum $\Sigma\tau_{i+k}\zeta_{i+k}$ does not change upon coalescence of any two segments. As regards the factor $\{(\zeta_i + \frac{1}{2}\tau_i)^2 - - (\zeta_{i-1} - \frac{1}{2}\zeta_{i-1})^2\}$, it does certainly not change upon coalescence of τ_{k-1} and τ_k when $k < i - 1$ or $k > i + 1$; neither does it change when $k = i - 1$ or $k = i + 1$, since in the first case the value of $\zeta_{i-1} - \frac{1}{2}\tau_{i-1}$, and in the second case the value of $\zeta_i + \frac{1}{2}\tau_i$ is unaffected. Finally, when $k = i$, the factor will disappear from the sum already just before coalescence has taken place, on account of the relation $\zeta_i + \frac{1}{2}\tau_i = \zeta_{i-1} - \frac{1}{2}\tau_{i-1}$, which is valid for $\lambda_i \to 0$; while after coalescence it is to be dropped altogether [6]).

We finally apply the method to find $d\varepsilon/dt$. If we multiply by S, we have:

$$S\,\varepsilon = N\,\overline{\tau_i^3}/12\,t^3.$$

In the case of coalescence of τ_i and τ_{i+1}, the sum of all τ_i^3 increases by $3(\tau_i^2\tau_{i+1} + \tau_i\tau_{i+1}{}^2)$. Since the frequency of this process is determined by the expression (61), it is easily found that

$$\frac{d\varepsilon}{dt} = -\frac{3\,\varepsilon}{t} + \frac{f_0}{8\,l\,t^4}\,\overline{(\tau_i + \tau_{i+1})^2\,\tau_i\tau_{i+1}}^{\,*} \tag{62}$$

the restricted mean value again referring to $\lambda_i \to 0$. An equivalent form is:

$$\frac{d}{dt}[l^2(1 + \omega^*)] = \frac{3\,f_0}{2\,l\,t}\,\overline{(\tau_i + \tau_{i+1})^2\,\tau_i\tau_{i+1}}^{\,*} \tag{63}$$

6) Another proof of the invariance of J_0 will be given in the third part of this paper (section **26**).

19. *Value of dl/dt.* — The expression (61), giving the number of cases in which segments coalesce, can be used to find the rate of decrease of N, which is given by:

$$dN/dt = -N\, f_0\, \overline{(\tau_i + \tau_{i+1})}^{\,*}/2t \tag{64}$$

In view of what has been observed in section 12 (end of second paragraph), we may replace $\frac{1}{2}\,\overline{(\tau_i+\tau_{i+1})}^{\,*}$ by $\overline{\tau_i}^{\,*}$. The symmetrical notation used in (64), however, has been retained on account of its analogy with (60) and (63).

Since $N = S/l$, S being a constant length, we find:

$$\frac{dl}{dt} = \frac{f_0\, l}{2t}\, \overline{\tau_i + \tau_{i+1}}^{\,*} \tag{65}$$

The restricted mean value refers to $\lambda_i \to 0$.

We can apply this result to eliminate dl/dt from equations (60), (63) and (59). This gives

$$\frac{d}{dt}(1+\omega) = \frac{f_0}{t}\left\{ \frac{\overline{(\tau_i+\tau_{i+1})\,\tau_i\,\tau_{i+1}}^{\,*}}{l^2} - (1+\omega)\, \frac{\overline{\tau_i+\tau_{i+1}}^{\,*}}{2} \right\} \tag{66}$$

$$\frac{d}{dt}(1+\omega^*) = \frac{f_0}{t}\left\{ \frac{3\,\overline{(\tau_i+\tau_{i+1})^2\,\tau_i\,\tau_{i+1}}^{\,*}}{2\,l^3} - (1+\omega^*)\, \overline{(\tau_i+\tau_{i+1})}^{\,*} \right\} \tag{67}$$

$$\frac{d}{dt}\left(\tilde{\omega} + \frac{1+\omega^*}{12}\right) = \frac{2\tilde{\omega}}{t} - \frac{f_0}{t}\left(\tilde{\omega} + \frac{1+\omega^*}{12}\right) \overline{(\tau_i + \tau_{i+1})}^{\,*} \tag{68}$$

These equations bring to light a few of the features of the system which determine the change with time of its statistical character. For instance, if we should start with a case in which all τ_i originally had the same value, equal to l (so that ω would be zero), while the length of the segments λ_i was distributed at random, equation (66) would give:

$$d(1+\omega)/dt = f_0\, l/t.$$

This would mean that an increase of the ratio $\overline{\tau_i^2}/l^2$ would immediately set in, which is, of course, to be expected.

It may be supposed that a statistical state can be reached with a pattern independent of the time. This supposition is made by those authors who speak of "self-preserving" correlation functions. For such a state eqs. (66) — (68) should have zero right hand members. We shall come back to this subject in section 24.

(To be continued).