MATHEMATICS

NEGATIONLESS INTUITIONISTIC MATHEMATICS II

BY

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(Communicated by Prof. L. E. J. BROUWER at the meeting of March 25, 1950)

In 1944 I gave a sketch of some parts of negationless intuitionistic mathematics in these Proceedings; afterwards I started on a more complete and systematic treatment [1]). This note is a sequal to it. As in the meantime, however, many remarks and objections reached me, I preface this note by a concise exposition of my point of view and some explanations to the second note.

In 1947 Prof. L. E. J. BROUWER gave a formulation of the directives of intuitionistic mathematics [2]). It is remarkable that negation does not occur in an explicit way, so one might be inclined to believe negationless mathematics to be a consequence of this formulation. The notion of species, however, is introduced in this way (translated from the Dutch text): "Finally in this construction of mathematics at any stage properties that can be supposed to hold for mathematical conceivabilities already obtained are allowed to be added as new mathematical conceivabilities under the name of species". By this formulation it is possible that there are properties that can be supposed to hold for mathematical conceivabilities already obtained but that are not known to be true. With it negation and null-species are introduced simultaneously but at the cost of evidence. Whatever are the properties that can be supposed? What other criterion could there be than 'to hold for mathematical conceivabilities already obtained'? In the definition of the notion of species the words "can be supposed" should be replaced by "are known". *One should restrict oneself in intuitionistic mathematics to mathematical conceivabilities and properties of those mathematical conceivabilities and one should not make suppositions of which one does not know whether it is possible to fulfil them.* (The well-known turn in mathematics: "Suppose ABC to be rectangular" seems to be a supposition, but mostly means: "Consider a rectangular triangle ABC").

ad § 1. 1. After the introduction of the natural numbers 1, 2, 3 the

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[1]) G. F. C. GRISS, Negatieloze intuïtionistische wiskunde. Versl. Ned. Akad. v. Wetensch., **53**, (1944).

Negationless intuitionistic mathematics. Proc. Kon. Ned. Akad. v. Wetensch., **49**, (1946).

[2]) L. E. J. BROUWER, Richtlijnen der intuïtionistische wiskunde. Proc. Kon. Ned. Akad. v. Wetensch., **50**, (1947).

natural number n' next to the natural number n was introduced by means of induction as follows:

"If, in this way, we have proceeded to E_n $(1, 2, \ldots, n)$, we can again imagine an element n', remaining the same, $n' = n'$, and distinguishable from each element p of E_n $(1, 2, \ldots, n)$, in formula $n' \neq p$, $p \neq n'$. They form the set $E_{n'}$ $(1, 2, \ldots, n')$."

$E_{n'}$ is called the sum of E_n and n', in other words: An element of $E_{n'}$ belongs to E_n or is n'. In this way the disjunction is defined in a particular case. It is evident the disjunction a or b in the usual meaning (the assertion a is true or the assertion b is true), does not occur in negationless mathematics, because there is no question of assertions that are not true. In general our definition of disjunction runs as follows: *a or b is true for all elements of the set V means that the property a holds for a subspecies V' and property b holds for a subspecies V'', V being the sum of V' and V''.*

ad § 1. 2. In accordance with the construction of natural numbers the proofs of properties of those numbers are always given by means of induction, until a system of properties is found, that can serve as a starting point of an axiomatic theory. At the time I used the disjunction in the proofs of the two properties concerning the relations of identity and distinguishability. Now we will show, how it is possible to avoid the use of disjunction in accordance with the remark made ad § 1. 1. For that purpose I formulate the first property: *If b is an element of E_m $(1, 2, \ldots, m)$, then b together with the elements of E_m that are distinguishable from b form E_m.*

Proof: The property holds for E_2. Suppose the proof has proceeded to E_n.1) Consider first an element b of E_n. The elements of $E_{n'}$ that differ from b are n' and those elements of E_n that differ from b. The latter form together with b the set E_n, and E_n together with n' forms $E_{n'}$. 2) Now consider $b = n'$. In this case the elements differing from b form the set E_n, so together with b the set $E_{n'}$. So the property holds for the elements of E_n and for n', so for all elements of $E_{n'}$.

The avoiding of the disjunction has little influence on the proof of the second property.

If for the elements a and b of E_m holds: $a \neq c$ for each $c \neq b$, then $a = b$.

Proof: The property holds for E_2. Suppose the proof has proceeded to E_n. 1) If $b = n'$, then a is distinguishable from each element of E_n, so $a = n'$ and $a = b$. 2) b is element of E_n; choose $c = n'$, then also a is an element of E_n, so $a = b$. The proof has been delivered for all elements of E_n and for n', so for all elements of $E_{n'}$.

Continuation of § 1.

4. *The set A $(1, 2, \ldots)$.*

In consequence of its construction A is the sum of E_n and the species A_n consisting of those elements of A that are distinguishable from the elements of E_n. Suppose an element of A to be distinguishable from all elements of A_n, then it is contained in E_n; in the case it is distinguishable from all elements of E_n it is an element of A_n. We now again give the proofs of property VI and VII of nr 2.

Proof of VI: A is the sum of E_b and the elements of A that are distinguishable from those of E_b. E_b is the sum of b and the elements of E_b distinguishable from b, so A is the sum of b and the elements of A distinguishable from b.

Proof of VII: a is distinguishable from all elements of A_b thus is an element of E_b. Hence $a = b$.

For the definition and properties of the order-relation in the set A one should consult nr 3. The modifications to be made are only slight ones.

5. *Finite and denumerable infinite species.*

A property of a natural number defines the species of natural numbers having this property. The species may consist of a single element a, e.g. the species of natural numbers identical with a. From the definition it follows that the species contains at least one element, in accordance with the remark made at the beginning. In particular every set is a species.

In the general theory of sets by BROUWER "verschieden" (different) means tacitly 'not the same'. Therefore this part of intuitionistic mathematics bears a paramount negative character.

A species A is said to be in one-to-one correspondance with a species B if to each element of A corresponds a single element of B, while of each element of B it is known to what element of A it corresponds. Then also B is in one-to-one correspondance with A. To two distinguishable elements of A correspond two distinguishable elements of B, and reversely. A and B are said to be of the same cardinal. Two one-to-one correspondances applied one after another give again a one-to-one correspondance.

A species A is mapped uniquely onto a species B, if to each element of A corresponds a single element of B. Then two distinguishable elements of B correspond to distinguishable elements of A.

Property: *If E_p and E_q have the same cardinal, then $p = q$.*

Proof: The property holds for $p = 2$. Suppose the proof has proceeded to n, that means from a one-to-one correspondance of E_n onto E_r it follows $r = n$, while a certain one-to-one correspondance of $E_{n'}$ onto $E_{m'}$ is given. If m' corresponds to n' then E_m and E_n have the same cardinal, so $m = n$, thus $m' = n'$.

If n' corresponds to the element a of E_m and if m' corresponds to b of E_n, then we replace the one-to-one correspondance by another one in which m' corresponds to n' and a to b. Then again E_m and E_n have the same cardinal, thus $m = n$, so $m' = n'$.

We now define: A finite species, of cardinal n, is a species having the same cardinal as E_n. The cardinal n of a finite species E is distinguishable from, less than or greater than the cardinal m of E', according to $n \neq m$, $n < m$, $n > m$.

A denumerable infinite species is a species having the same cardinal as the set of natural numbers.

Property: *If the finite species E has the same cardinal as a proper subspecies of the finite species E', then the cardinal of E is less than that of E'.*

Proof: Suppose E has the cardinal n, then also a proper subspecies E'' of E' has the cardinal n. Now we map the species E' in such a way onto E_m, that E'' is mapped onto E_n. E' contains an element that is distinguishable from each element of E''. This element now is mapped onto an element of E_m, that is distinguishable from each element of E_n.

Likewise the reverse holds: *In the case the cardinal of a finite species E is less than that of a finite species E', E has the same cardinal as a proper subspecies of E'.*

Also: If E is a finite subspecies of the set A of natural numbers, then it is possible to determine a number m, such that E is a subspecies of E_m; so E is a proper subspecies of A.

Each finite species has the same cardinal as a proper subspecies of a denumerable infinite species.

A species containing at least three mutually distinguishable elements is called ordered, if there exists between elements a and b of the species a relation $a < b$ $(b > a)$, satisfying the following conditions:

$$O_1 \quad a \neq b \rightarrow a < b \text{ or } b < a$$
$$O_2 \quad a < b \rightarrow a \neq b$$
$$O_3 \quad a < b \text{ and } b < c \rightarrow a < c.$$

(The second condition replaces the negative condition $a = b$, $a < b$ and $b < a$ exclude one another).

One easily proves by induction:

A finite ordered species contains a first and a final element. Two finite ordered species having the same cardinal number have the same ordinal number.

6. *The fundamental operations.*

We mention only.

Definition of addition: Adding 1 is a transformation that to each number let correspond its successor; adding n' means adding n and then 1, in formula $n + 1 = n'$ and $a + (n + 1) = (a + n) + 1$.

The transformation is unique:

$$a = b \rightarrow a + c = b + c \text{ and } a + c \neq b + c \rightarrow a \neq b.$$

Properties:

$$A_1 \quad a + (b + c) = (a + b) + c$$
$$A_2 \quad a + b = b + a$$
$$A_3 \quad a \neq b \rightarrow a + c \neq b + c.$$

These are the axioms of any commutative group. Characteristic for natural numbers is:

$$N \quad a + b \neq 1$$

and for the order-relation

$$O_4 \quad a > b \rightarrow a + c > b + c.$$

All these properties can easily be proved with induction. Further properties can be deduced from them, without using induction in an explicit way.

Property: $a + b \neq a$.

Proof: $c + b \neq 1 \rightarrow 1 + b \neq 1 \rightarrow a + 1 + b \neq a + 1 \rightarrow a + b \neq a$ (N, A_1, A_2, A_3).

Property: $a + c = b + c \rightarrow a = b$.

Proof: Take $d \neq b$, then $d + c \neq b + c \rightarrow d + c \neq a + c \rightarrow d \neq a$. So $a \neq d$ for any $d \neq b$, thus $a = b$ (A_3, VI).

Property: $a + b > a$.

Proof: $b \geqq 1 \rightarrow a + b \geqq a + 1 \rightarrow a + b > a$ (nr 3, O_4).

Property: $a + d > b + d \rightarrow a > b$.

Proof: $a + d > b + d \rightarrow a + d \neq b + d \rightarrow a \neq b$. Take $c < b$, then $c + d < b + d < a + d \rightarrow c + d \neq a + d \rightarrow c \neq a$. $a \neq b$ and $c \neq a$ for each $c < b \rightarrow a > b$ (nr 3).

Definition: Multiplication is a unique transformation defined by

$$a \times 1 = a \text{ and } a\,(n + 1) = an + a,$$

so $a = b \rightarrow ac = bc$ and $ac \neq bc \rightarrow a \neq b$.

Properties:

$$M_1 \quad a(bc) = (ab)c.$$
$$M_2 \quad ab = ba.$$
$$M_3 \quad a(b + c) = ab + ac.$$
$$M_4 \quad a \neq b \rightarrow ac \neq bc.$$
$$O_5 \quad a > b \rightarrow ac > bc.$$

These properties, again, can easily be proved by means of induction. Further

$$ac = bc \rightarrow a = b.$$
$$ac > bc \rightarrow a > b.$$

Property: $z \neq 1 \rightarrow xz \neq 1$.

Proof: $z \neq 1 \rightarrow z > 1 \rightarrow xz > x \rightarrow xz > 1 \rightarrow xz \neq 1$ (nr 3, O_5).

Property: $xy = 1 \rightarrow x = 1$ and $y = 1$.

Proof: Take $z \neq 1 \rightarrow xz \neq 1 \rightarrow xz \neq xy \rightarrow z \neq y$. So $z \neq y$ for any $z \neq 1 \rightarrow y = 1$. Then $x = 1$.

Definition of subtraction. If for three numbers a, b and v holds $v + a = b$, v is called the difference between b and a.

Definition: If $n \neq 1$ we mean by 'n diminishing by 1': determining the predecessor $'n$ (nr 3); $'n = n - 1$. If $a < b$ and $a \neq 1$ we mean by 'b diminishing by a': diminishing by 1 and after that by $'a$; $b - a = {}'b - {}'a$.

Property: If $b > a$, $b - a$ is the difference between b and a.

Proof: $(b - 1) + 1 = {}'b + 1 = b$. Suppose the proof has proceeded to $'a$. $(b - a) + a = ('b - {}'a) + ('a + 1) = \{('b - {}'a) + {}'a\} + 1 = {}'b + 1 = b$.

Property: There is only one difference between b and a $(b > a)$.

Proof: If $v + a = b$ and $w + a = b$, then $v + a = w + a$, so $v = w$. We first proved that if $b > a$ at least one value can be found being the difference $b - a$. It need not be told separately that no such value exists in the case $b \leqq a$, for if a difference $b - a$ exists, then $v + a = b$ thus $b > a$ and then $b \neq c$ for each $c \leqq a$.

That there exists at most one value for the difference $b - a$ does not mean that there are no two such values but that one always finds the same value.

Definition of division: If for three numbers a, b and q holds $q\,a = b$, q is called the quotient of b and a.

Property: If a and b have a quotient, there is only one quotient.

Property: If $a \leqq b$ $(a \neq 1)$ two numbers q and $r < a$ can be determined in such a way that $b = qa$ or $b = qa + r$.

Proof: If $b = a$, then $b = 1.\ a$. Let the proof have proceeded to $b = n > a$, then $n = qa$ or $n = qa + r$ $(r < a)$. So
$n' = qa + 1$ $(1 < a)$ or $n' = qa + r'$ with $r' < a$ or $r' < a$; in the latter case $n' = qa + a = (q + 1)a$.

On this is founded EUCLIDES' algorithm and the theory of division and separation in primefactors which I only give in outline.

Definition: a and b are called mutually divisible if it is possible to determine d $\neq 1$, p and q such that $a = pd$ and $b = qd$; a and b are called mutually prime if for each $c \neq 1$, p and q holds $a \neq pc$ or $b \neq qc$.

Property: Two numbers are either mutually prime or mutually divisible.

Property: If a and b are mutually prime and c and d are mutually divisible then (a, b) is distinguishable from (c, d).

Definition: a is divisible if it is possible to determine $b \neq 1$ and $c \neq 1$ in such a way that $a = bc$; a is prime if $a \neq bc$ for each $b \neq 1$ and $c \neq 1$.

Property: A number is either prime or divisible.

Property: A divisible numbers can be factorized in a unique way into primefactors.

Property: Each prime number has an immediately succeeding prime number.

§ 2. *The rational number.*

1. *The number* 0.

We add to the set A $(1, 2, \ldots)$ of natural numbers the element 0, distinguishable from all elements of A. How has that to be done?

One may interrupt the enumeration of the elements of A and intersperse the number 0. In the case no operations and orderrelations have been introduced, this will bring forth nothing new. We define however,

$$0 < a, \quad a + 0 = 0 + a = a \quad \text{and} \quad a . 0 = 0 . a = 0.$$

One might also begin with 0 and add the set of natural numbers. Properly speaking one has to begin all over again and that with $(0, 1, 2, \ldots)$, define the fundamental operations and order-relations, noticing the subset $(1, 2, \ldots)$ is in one-to-one accordance with the set of natural numbers with preservation of fundamental operations and order-relations. The properties with proofs concerning order-relations and operations of natural numbers remain valid with slight modifications. It is evident the number 0 has nothing to do with a cardinalnumber. (no more than the negative number -1 or the fraction $\frac{1}{2}$ leads to a cardinal number).

2. *Entire numbers.*

Introduce in accordance with nr 1 $(0, 1, -1, 2, -2, \ldots)$.

Order relations: $0 < 1$, $-1 < 0$, $-1 < 1$, $n' >$ each element of $(0, 1, \ldots, -n)$ and $-n' <$ each element of $(0, 1, \ldots, n')$.

The further theory of entire numbers does not offer any difficulty.

Negative numbers have as little to do with negation as the number 0 with cardinal numbers.

3. *Rational numbers.*

I will treat the theory of rational numbers more fully as it has to serve as a foundation of the theory of real numbers. If one defines a fraction as a pair of entire numbers the fractions $^2/_3$ and $^3/_2$ ought to be identified. One distinguishes them, however, tacitly by their position. If one defines more precisely by calling an 'ordered' pair of entire numbers a fraction, it is not yet evident how to distinguish $^2/_2$ from 2. One must be able to distinguish in each pair the two numbers, e.g. by indices 1 and 2 or in principle more simply, by writing the numerator in arabic figures and the denominator in roman figures. We choose the manner of writing 5_1 and 4_2, where these symbols are to be considered as one sign.

Definition: We construct a series of natural numbers 1_2, 2_2 etc. and a series of entire numbers 0_1, -1_1, 1_1, etc. where each element of the second series is distinguishable from each element of the first series. Each pair of numbers originating by choosing one number from the first and another one from the second series is called a fraction. The number of the first series is called denumerator, that from the second one numerator.

Two fractions (a_1, b_2) and (c_1, d_2) are called the same if $a_1 = c_1$ and $b_2 = d_2$; they are distinguishable if at least $a_1 \neq c_1$ or $b_2 \neq d_2$. This definition is in accordance with a general definition of identity and distinguishability that will be given later on.

Much more important is the notion equal and different for fractions.

For this purpose we first arrange to calculate with numerators and denumerators by suppressing the indices. We shall denote the relations of identity and distinguishability by $\equiv$ and $\not\equiv$, respectively and the relations of equality and difference by $=$ and $\neq$ and define

$$(a_1\, b_2) = (c_1\, d_2) \text{ if } ad \equiv bc$$
$$(a_1\, b_2) \neq (e_1\, f_2) \text{ if } af \not\equiv be.$$

These relations satisfy the properties I — VII of § 1. 2.

We give the proofs for V and VI.

V. $(a_1\, b_2) = (c_1\, d_2)$ and $(c_1\, d_2) \neq (e_1\, f_2) \to ad \equiv bc$ and $ef \not\equiv ed \to adf \equiv$ $\equiv ebf$ and $cbf \not\equiv ebd \to adf \not\equiv ebd \to af \not\equiv eb \to (a_1\, b_2) \neq (e_1\, f_2)$.

VI. $(a_1\, b_2) \neq (p_1\, q_2)$ for each $(p_1\, q_2) \neq (c_1\, d_2) \to$
$aq \not\equiv bp$ for each p and q for that $pd \not\equiv pc \to$ (for $q \equiv db$)
$adb \not\equiv bp$ for each p for that $pd \not\equiv dbc \to$
$ad \not\equiv p$ for each $p \not\equiv bc \to ad \equiv bc \to (a_1\, b_2) = (c_1\, d_2)$.

We shall always call the relations $\alpha = \beta$ and $\alpha \neq \gamma$ satisfying the conditions I — V relations of equality and difference, if $\alpha \equiv \beta \to \alpha = \beta$ and $\alpha \neq \gamma \to \alpha \not\equiv \gamma$. Sometimes one means by a 'fraction' the species of all mutually equal fractions. In virtue of the developments in § 4 of my first note on negationless mathematics, for 'fractions' an identity-relation and distinguishability-relation holds satisfying conditions I — VII.