MATHEMATICS

NOTE ON THE LOCATION OF ZEROS OF POLYNOMIALS

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GYULA DE SZ. NAGY proved some theorems on the location of zeros of polynomials [1]. These theorems are generalisations of results obtained by G. SZEGÖ [2].

The present note deals with similar problems (Theorems I, II, III and IV). Theorem V is a generalisation of a theorem of the first mentioned author, see [3]. We prove the following

Theorem I.

Let the polynomial

$$f(z) = (z-a_1)(z-a_2)\dots(z-a_n)$$

have all its zeros in the circle $|z-a| \leqslant R$, *and let the polynomial*

$$g(z) = (z-b_1)(z-b_2)\dots(z-b_n)$$

have no zero in the circle $|z-a| \leqslant \varrho$, $\varrho < R$.

Then no polynomial

$$h(z) = \lambda f(z) - g(z), \quad |\lambda| \leqslant t^n, \quad 0 \leqslant t < \frac{\varrho}{R},$$

has a zero in the circle

(1) $$|z-a| \leqslant r = \frac{\varrho - Rt}{1+t} \; (\leqslant \varrho).$$

Proof.

Every point z_0 with $h(z_0) \neq 0$ satisfies

$$\left|\frac{g(z_0)}{f(z_0)}\right| \neq |\lambda|.$$

Now we have for every point z_0 of the region (1)

$$|z_0 - a_k| \leqslant |z_0 - a| + |a_k - a| \leqslant r + R,$$

and

$$|z_0 - b_k| \geqslant |b_k - a| - |z_0 - a| > \varrho - r,$$

so that

$$\left|\frac{f(z_0)}{g(z_0)}\right| = \prod_{k=1}^{n} \left|\frac{z_0 - a_k}{z_0 - b_k}\right| < \left(\frac{r+R}{\varrho - r}\right)^n = \frac{1}{t^n} \leqslant \frac{1}{|\lambda|}$$

This completes the proof of Theorem I.

A generalisation of Theorem I is

Theorem II.

Let the polynomial

$$f(z) = (z-a_1)(z-a_2)\dots(z-a_n)$$

have all its zeros in the circle $|z-\alpha| \leqslant \varrho_1$, *and let the polynomial*

$$g(z) = (z-b_1)(z-b_2)\dots(z-b_n)$$

have no zero in the circle

$$|z-\beta| \leqslant \varrho_2,\ \varrho_1 > \varrho_2,\ |\alpha-\beta| < \min(\varrho_1-\varrho_2, \varrho_2).$$

Then no polynomial

$$h(z) = \lambda f(z) - g(z),\ |\lambda|^{1/n} < \frac{\varrho_2 - |\beta-\alpha|}{\varrho_1},$$

has a zero in the region

$$(2) \qquad |\lambda|^{1/n}\cdot|z-\alpha| + |z-\beta| \leqslant \varrho_2 - \varrho_1|\lambda|^{1/n}.$$

Proof. We put $|z_0-\alpha| = r_1$ and $|z_0-\beta| = r_2$, where z_0 is a point of the region (2). From our assumptions it follows that z_0 is an interior point of the circle $|z-\beta| \leqslant \varrho_2$.

Now we have

$$|z_0-a_k| \leqslant |z_0-\alpha| + |a_k-\alpha| \leqslant r_1 + \varrho_1,$$

and

$$|z_0-b_k| \geqslant |b_k-\beta| - |z_0-\beta| > \varrho_2 - r_2,$$

so that

$$\left|\frac{z_0-a_k}{z_0-b_k}\right| < \frac{r_1+\varrho_1}{\varrho_2-r_2}, \text{ and } \left|\frac{f(z_0)}{g(z_0)}\right| < \left(\frac{r_1+\varrho_1}{\varrho_2-r_2}\right)^n \leqslant \frac{1}{|\lambda|},$$

and from these last inequalities it follows that z_0 is not a zero of $h(z)$.

Theorem III.

Let G_1 *be a half-plane containing the zeros of the polynomial*

$$f(z) = (z-a_1)(z-a_2)\dots(z-a_n).$$

Let G_2 *be the complementary half-plane containing the zeros of the polynomial*

$$g(z) = (z-b_1)(z-b_2)\dots(z-b_n),$$

and let l *be the boundary line of* G_1 *and* G_2.

Let α *and* β *be two points of* G_1 *with*

$$|\alpha-\beta| > d, \text{ where } d = \max_{k=1,\dots,n} |\alpha-a_k|.$$

Let P *be the parabola with* β *as focus and with* l *as directrix, and let* H

be the hyperbola with foci at a and β, and with the major axis d. Then the polynomial

$$h(z) = f(z) + g(z)$$

has no zero in the region R, the common part of the interior of P, and the interior of H containing a.

Remarks. 1. The set of points from which no tangent can be drawn to a given conic section is said to be the interior of that conic section.

2. If we choose β on the perpendicular from a on l, then the set of points R is not empty.

Proof. We denote by z_0 an arbitrary point of R.

It is obvious, that

$$|z_0 - b_k| > |z_0 - \beta|,$$

and

$$|z_0 - a_k| \leqslant |z_0 - a| + |a_k - a| \leqslant |z_0 - a| + d.$$

Hence

$$\left|\frac{f(z_0)}{g(z_0)}\right| = \prod_{k=1}^{n} \left|\frac{z_0 - a_k}{z_0 - b_k}\right| < \left(\frac{|z_0 - a| + d}{|z_0 - \beta|}\right)^n < 1, \tag{3}$$

where the last inequality follows from

$$|z_0 - \beta| - |z_0 - a| > d$$

From (3) we see that z_0 is no zero of $h(z)$.

Theorem IV.

Let G_1 be a half-plane containing the zeros of the polynomial

$$f(z) = (z - a_1)(z - a_2) \ldots (z - a_n).$$

Let G_2 be the complementary half-plane containing the points a and α, and let l be the boundary line of G_1 and G_2.

Let P be the parabola with a as focus and l as directrix. Let C be the circle of Apollonius for the points a and α, with ratio t $(0 < t < 1)$, and of which a is an interior point (this means that C is the set of points z with

$$|z - a| = t\,|z - \alpha|).$$

Let $g(z)$ be the polynomial

$$g(z) = (z - a)^n.$$

Then no polynomial

$$h(z) = g(z) + \lambda f(z),\ |\lambda| = t^n,\ 0 < t < 1,$$

has a zero in the region R, which the interiors of P and C have in common.

Proof. We denote by z_0 an arbitrary point of R.

Then

$$|z_0 - a_k| > |z_0 - a|, \text{ and } |z_0 - a| < t\,|z_0 - \alpha|,$$

so that

$$\left|\frac{g(z_0)}{f(z_0)}\right| = \prod_{k=1}^{n} \left|\frac{z_0 - a}{z_0 - a_k}\right| < \left|\frac{z_0 - a}{z_0 - \alpha}\right|^n < t^n \leqslant |\lambda|.$$

Hence

$$|g(z_0)| < |\lambda f(z_0)|.$$

This means that z_0 is no zero of $h(z)$.

Theorem V.

Let

$$f(z) = (z - a_1)(z - a_2) \dots (z - a_n),$$

with

$$a_k = x_k + i y_k \qquad (k = 1, \dots, n),$$

$$g(z) = (z - b_1)(z - b_2) \dots (z - b_n),$$

with

$$b_k = x_k + i y_k^*, \; y_k > y_k^* \qquad (k = 1, \dots, n),$$

$$F(z) = \frac{f(z)}{g(z)} = \frac{(z - a_1)(z - a_2) \dots (z - a_n)}{(z - b_1)(z - b_2) \dots (z - b_n)},$$

$$F'(\zeta) = 0, \; F(\zeta) \neq 0, \; \zeta = \xi + i\eta.$$

Then:

(*) *If the intervals*

$$y_k \geqslant y \geqslant y_k^* \qquad (k = 1, 2, \dots, n) \tag{4}$$

have an interval I in common, η is not an interior point of I.

(**) *ζ is not a point of the region R, which the interiors of the hyperbolas H_k*

$$(x - x_k)^2 - (y - y_k)(y - y_k^*) = 0 \qquad (k = 1, \dots, n)$$

have in common. ζ is an interior point of one H_k at least, and an exterior point of one the other H_k at least.

(***) *If* $(k = 1, \dots, n)$

$$2 m_k = y_k + y_k^*,$$

then ζ is not a point of the regions R_1 and R_2, the common parts of the regions

$$(x - x_k)(y - m_k) > 0,$$

and

$$(x - x_k)(y - m_k) < 0,$$

respectively.

Proof.

$$\frac{F'(z)}{F(z)} = \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)} = \sum_{k=1}^{n} \left\{ \frac{1}{z - a_k} - \frac{1}{z - b_k} \right\} =$$

$$i \sum_{k=1}^{n} \frac{y_k - y_k^*}{\{x - x_k + i(y - y_k)\}\{x - x_k + i(y - y_k^*)\}} =$$

$$i \sum_{k=1}^{n} \frac{y_k - y_k^*}{\{(x - x_k)^2 - (y - y_k)(y - y_k^*)\} + i\{2xy - x y_k - x y_k^* - 2 y x_k + x_k y_k + x_k y_k^*\}}$$

If now $\zeta = \xi + i\eta$, $F'(\zeta) = 0$ and $F(\zeta) \neq 0$, then

$$\frac{F'(\zeta)}{F(\zeta)} = i \sum_{k=1}^{n} \frac{y_k - \overset{*}{y_k}}{A_k + i B_k} = \sum_{k=1}^{n} \frac{B_k (y_k - \overset{*}{y_k})}{A_k^2 + B_k^2} + i \sum_{k=1}^{n} \frac{A_k (y_k - \overset{*}{y_k})}{A_k^2 + B_k^2} = 0,$$

$$\text{(5)} \qquad \sum_{k=1}^{n} \frac{A_k (y_k - \overset{*}{y_k})}{A_k^2 + B_k^2} = 0, \quad \sum_{k=1}^{n} \frac{B_k (y_k - \overset{*}{y_k})}{A_k^2 + B_k^2} = 0,$$

with

$$A_k = (\xi - x_k)^2 - (\eta - y_k)(\eta - \overset{*}{y_k}),$$

and

$$B_k = 2\xi\eta - \xi y_k - \xi \overset{*}{y_k} - 2\eta x_k + x_k y_k + x_k \overset{*}{y_k}.$$

If ζ is a point of R, then the inequalities

$$\text{(6)} \qquad A_k > 0 \qquad (k = 1, \ldots, n)$$

hold. From (6) and our assumption $y_k - \overset{*}{y_k} > 0$ $(k = 1, \ldots, n)$ it follows that the first equality of (5) does not hold.

So (**) is proved.

(*) is an immediate consequence of (**).

Furthermore we have

$$\sum_{k=1}^{n} \frac{B_k (y_k - \overset{*}{y_k})}{A_k^2 + B_k^2} = \sum_{k=1}^{n} \frac{(y_k - \overset{*}{y_k})(\xi - x_k)(2\eta - y_k - \overset{*}{y_k})}{A_k^2 + B_k^2} =$$

$$2 \sum_{k=1}^{n} \frac{(y_k - \overset{*}{y_k})(\xi - x_k)(\eta - m_k)}{A_k^2 + B_k^2}$$

It can easily be seen that the last expression does not vanish if ζ is a point of R_1, or R_2. This concludes the proof of (***).

LITERATURE

[1] GYULA DE SZ. NAGY, Generalisations of certain theorems of G. SZEGÖ on the location of zeros of polynomials, Bull. Am. Math. Soc., 53, 12, 1164—1169 (1947).

[2] SZEGÖ, G., Bemerkungen zu einem Satz von J. H. GRACE über die Wurzeln algebraischer Gleichungen, Math. Zeitschr. 13, 28—55 (1922).

[3] GYULA DE SZ. NAGY, Ueber rationale Funktionen, deren Nulstellen und Pole an entgegengesetzten Seiten einer Geraden liegen, Hung. Acta Math., 1 12—16 (1949).