CORRELATION PROBLEMS IN A ONE-DIMENSIONAL MODEL OF TURBULENCE. III. *)

BY

J. M. BURGERS

(*Mededeling no. 65c uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft*)

(Communicated at the meeting of April 29, 1950)

20. *Further investigation of the correlation function* $\overline{v_1 v_2}$. — In Part II expressions have been constructed for the correlation functions $\overline{v_1 v_2}$ and $\overline{v_1^2 v_2}$. These expressions have been applied to investigate some consequences of the fundamental equation (12), particular attention being given to values of η small in comparison with the mean length l of the segments λ_i. In the results an important part was played by the statistical parameter $f_0 = \lim f_1(\lambda)$ for $\lambda \to 0$, together with certain mean values referring to $\lambda \to 0$.

We now turn to some long distance relations. In the first place we will show that the expression (47) for $\overline{v_1 v_2}$ and its derivative (48) vanish for large values of η.

We write

$$\int_0^\eta d\lambda\, f_k(\lambda) = g_k \tag{69}$$

so that g_k measures the probability for Λ_k to be less than η [Λ_k is the sum of the lengths of k consecutive segments λ_i, from λ_{i+1} to λ_{i+k} inclusive; compare (37)]. It will be evident that the expression:

$$l^2 \sum_{k=1}^{\infty} \varphi_k = \int_0^\eta d\lambda \sum_1^\infty f_k(\lambda)\, \overline{\tau_i \tau_{i+k}}^{\,*} \tag{70}$$

combines all cases defined by the inequalities

$$\begin{aligned} 0 &< \lambda_{i+1} && < \eta \\ 0 &< \lambda_{i+1} + \lambda_{i+2} && < \eta \\ 0 &< \lambda_{i+1} + \lambda_{i+2} + \lambda_{i+3} && < \eta, \text{ etc.,} \end{aligned}$$

each case with its proper frequency of occurrence. Without omitting any one we can re-arrange these cases in such a way that we first consider the class for which

$$\lambda_{i+1} < \eta < \lambda_{i+1} + \lambda_{i+2}\,;$$

*) Continued from these Proceedings, p. 393—406. — In eq. (50), p. 401, the last factor should be η^2/l^2, the same as in eq. (49*b*).

next the class for which

$$\lambda_{i+1}+\lambda_{i+2}<\eta<\lambda_{i+1}+\lambda_{i+2}+\lambda_{i+3}$$

(which naturally entails $\lambda_{i+1}<\eta$); then the class for which

$$\lambda_{i+1}+\lambda_{i+2}+\lambda_{i+3}<\eta<\lambda_{i+1}+\lambda_{i+2}+\lambda_{i+3}+\lambda_{i+4}$$

(which entails $\lambda_{i+1}<\eta$; $\lambda_{i+1}+\lambda_{i+2}<\eta$), etc. The probabilities to be assigned to the classes thus distinguished (that is, the relative numbers of cases falling in any one of them) are given by

$$(A) \qquad g_1-g_2 \quad ; \quad g_2-g_3 \quad ; \quad g_3-g_4\,; \ \dots$$

It follows from (36) that

$$(B) \qquad \sum_1^\infty (g_k-g_{k+1})=1$$

if η is sufficiently large.

We take those terms of the sum (70) which refer to the particular class defined by

$$(C) \qquad \Lambda_k<\eta<\Lambda_{k+1}$$

and add together the quantities $\tau_i\tau_{i+1}$, $\tau_i\tau_{i+2}$, $\tau_i\tau_{i+3}$,... occurring in these terms. The contribution obtained in this way can be written:

$$(D) \qquad (g_k-g_{k+1})\,\overline{\tau_i T'_k}^{**}$$

where

$$(71) \qquad T'_k=\tau_{i+1}+\tau_{i+2}+\dots+\tau_{i+k}=T_k-\tfrac{1}{2}\tau_i+\tfrac{1}{2}\tau_{i+k}=\sigma_{i+k}-\sigma_i$$

[compare (44)], while the new type of restricted mean values refers to the cases satisfying the inequalities (C). When (D) is summed with respect to k from $k=1$ to infinity, we obtain the expression (70).

We write:

$$(72) \qquad T'_k=\eta-\tfrac{1}{2}\tau_i+\zeta_i+\theta$$

(compare fig. 7). Since $T_k+\zeta_{i+k}-\zeta_i=\Lambda_k$, so that

$$T'_k=\Lambda_k-\tfrac{1}{2}\tau_i+\zeta_i+\tfrac{1}{2}\tau_{i+k}-\zeta_{i+k},$$

we find:

$$\theta=\Lambda_k-\eta+\tfrac{1}{2}\tau_{i+k}-\zeta_{i+k}=\Lambda_{k+1}-\eta-\tfrac{1}{2}\tau_{i+k+1}-\zeta_{i+k+1}.$$

Hence in consequence of (C) θ satisfies the inequalities:

$$-\tfrac{1}{2}\tau_{i+k+1}-\zeta_{i+k+1}<\theta<\tfrac{1}{2}\tau_{i+k}-\zeta_{i+k}.$$

When η is sufficiently large, g_k will practically be equal to 1 and $g_k-g_{k+1}=0$, for all k which are small compared with η/l. Terms with small values of k consequently will play an insignificant part in the sum of the quantities (D). We can safely assume that in all significant terms θ will be completely independent of τ_i, so that

$$(72a) \qquad \overline{\tau_i\,\theta}^{**}=\overline{\tau_i}\cdot\overline{\theta}^{**}=0$$

For later use we add the formula:

$$\overline{\tau_i\,\theta^2}^{**} = \overline{\tau_i}\cdot\overline{\theta^2}^{**} = \tfrac{1}{3}\,\overline{\{(\tfrac{1}{2}\,\tau_{i+k+1}+\zeta_{i+k+1})^3+(\tfrac{1}{2}\,\tau_{i+k}-\zeta_{i+k})^3\}}^{**},$$

which for large η we can assume to become equal to its ordinary mean value, so that

$$(72b)\qquad \overline{\tau_i\,\theta^2}^{**} = \overline{\tau_i\,\zeta_i^2} + \tfrac{1}{12}\,\overline{\tau_i^3} = l^3\,\{\tilde{\omega} + \tfrac{1}{12}\,(1+\omega^*)\}$$

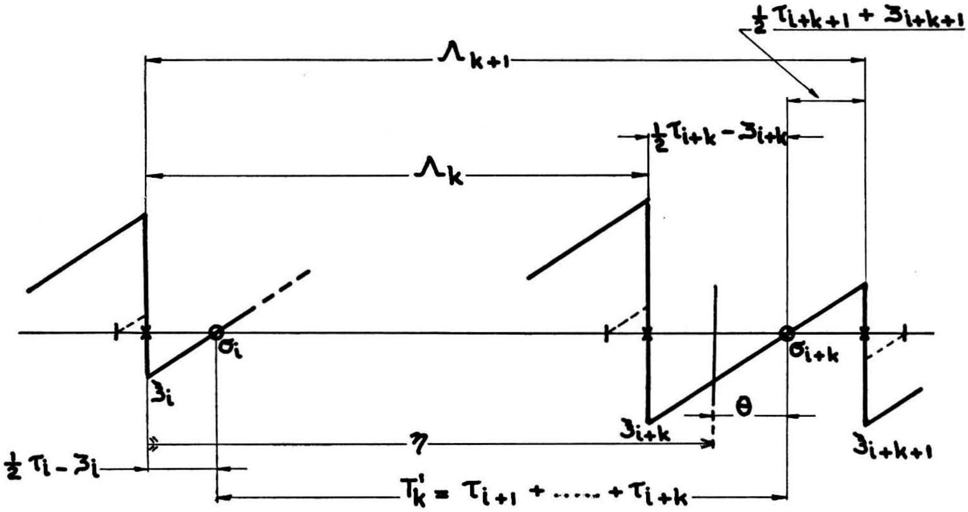

Fig. 7.

Since $\overline{\tau_i\,\zeta_i}^{**}$ will be zero in the same way as $\overline{\tau_i\,\zeta_i}$, the only terms remaining in the quantity (D) are:

$$(g_k - g_{k+1})\,\overline{\tau_i\,(\eta-\tfrac{1}{2}\,\tau_i)}^{**}.$$

Having regard to (B), we thus arrive at the result:

$$(73)\qquad \sum_{k=1}^{\infty}\varphi_k \cong \frac{\overline{\tau_i\,(\eta-\frac{1}{2}\,\tau_i)}}{l^2} = \frac{\eta}{l} - \tfrac{1}{2}\,(1+\omega)\qquad (\text{for } \eta\to\infty)$$

In connection with (48) this already proves that $\partial(\overline{v_1 v_2})/\partial\eta$ vanishes for $\eta\to\infty$.

Differentiation of (73) with respect to η gives:

$$(74)\qquad \sum_{k=1}^{\infty} f_k(\eta)\,\overline{\tau_i\,\tau_{i+k}}^{*} \cong l\qquad (\text{for } \eta\to\infty)$$

Since terms with small k do not contribute materially to this sum when η is large, we can safely replace $\overline{\tau_i\,\tau_{i+k}}^{*}$ by l^2, so that we find:

$$(74a)\qquad \sum_{k=1}^{\infty} f_k(\eta) \cong 1/l\qquad (\text{for } \eta\to\infty)$$

which result is identical with eq. $(39a)$.

By means of a similar re-arrangement as was used in the case of (70) we obtain the transformation:

$$l^3 \sum_{k=1}^{\infty} \Phi_k = \tfrac{1}{2} \sum (g_k - g_{k+1}) \, \overline{(\tau_i^2 T_k' + \tau_i T_k'^2)}^{**} .$$

Expressing T_k' through η and θ as before and making use of (72*a*) and (72*b*), we obtain, after some calculation:

$$\sum_{k=1}^{\infty} \Phi_k \cong \frac{\eta^2}{2\,l^2} + \tilde{\omega} - \tfrac{1}{12}\,(1+\omega^*) \qquad (\text{for } \eta \to \infty) \tag{75}$$

Finally we consider:

$$l^3 \sum_{k=1}^{\infty} (\Phi_k - \chi_k) = \int_0^{\eta} d\lambda \sum_1^{\infty} f_k(\lambda) \, \overline{(T_k - \lambda)\, \tau_i \tau_{i+k}}^{*} ,$$

where λ stands for Λ_k. We have already seen that $T_k - \Lambda_k = \zeta_i - \zeta_{i+k}$; hence, making use of the second invariant property:

$$l^3 \sum_{k=1}^{\infty} (\Phi_k - \chi_k) = 2 \int_0^{\eta} d\lambda \sum_1^{\infty} f_k(\lambda) \, \overline{\zeta_i \tau_i \tau_{i+k}}^{*} \tag{76}$$

By re-arrangement this is transformed into:

$$2 \sum (g_k - g_{k+1}) \, \overline{\zeta_i \tau_i T_k'}^{**} ,$$

which, on working out, comes down to $2\, \overline{\tau_i \zeta_i^2}^{**}$, for which we can take $2\, \tilde{\omega}\, l^3$. Consequently:

$$\sum_{k=1}^{\infty} (\Phi_k - \chi_k) \cong 2\,\tilde{\omega} \qquad (\text{for } \eta \to \infty) \tag{77a}$$

and

$$\sum_{k=1}^{\infty} \chi_k \cong \frac{\eta^2}{2l^2} - \tilde{\omega} - \tfrac{1}{12}(1+\omega^*) \qquad (\text{for } \eta \to \infty) \tag{77b}$$

These results prove that both $\overline{v_1 v_2}$ and $\partial(\overline{v_1^2 v_2})/\partial\eta$ vanish for $\eta \to \infty$. I have not calculated $\sum X_k$, but it seems safe to assume that the expression for $\overline{v_1^2 v_2}$ likewise will vanish for $\eta \to \infty$. We shall make use of this assumption in the calculation of its FOURIER transform.

21. The following observations can be made in connection with eq. (73). We write:

$$\overline{\tau_i \tau_{i+k}} = l^2(1+\omega_k) \quad ; \quad \overline{\tau_i \tau_{i+k}}^{*} = l^2(1+\omega_k^*) \tag{78}$$

so that ω_k is a constant, whereas ω_k^* is a function of $\eta = \Lambda_k$, to which refers the restricted mean value [7]). It follows from (42) that

$$\int_0^{\infty} f_k(\eta)\, \omega_k^*\, d\eta = \omega_k .$$

[7]) The ω^* which was introduced in (31), has no connection with the ω_k^* defined here.

We can assume that ω_k and ω_k^* will vanish for large k. Hence for all relevant values of k we can write

$$\int_0^{\eta} f_k(\eta)\, \omega_k^*\, d\eta \cong \omega_k,$$

provided η has a large value. Equation (73) consequently gives:

$$\sum_{k=1}^{\infty} g_k(\eta) = \int_0^{\eta} \{\Sigma f_k(\eta)\}\, d\eta = \frac{\eta}{l} - \tfrac{1}{2} - (\tfrac{1}{2}\omega + \sum_1^{\infty} \omega_k) \tag{79}$$

for large values of η. This result confirms (40*a*) and gives an interpretation of the constant which had been left undetermined.

Both equation (72*a*) and equation (79) are connected in a certain way with the condition to be satisfied by the distribution of the λ_i and τ_i, in order that the mean value $\overline{\zeta_i^2}$ shall exist. Reference to this condition has been made in sections **8** and **12**. One way of satisfying (79) is to assume that

$$\sum_1^{\infty} g_k(\eta) = \frac{\eta}{l} - \tfrac{1}{2} \tag{80a}$$

and

$$\tfrac{1}{2}\omega + \sum_1^{\infty} \omega_k = 0 \tag{80b}$$

The first equation is obtained when we suppose that for large values of k the dispersion of Λ_k about its mean value $k\,l$ is symmetrical and independent of k. The system of functions g_k then obtains the character indicated in fig. 8. We can assume that $g_k\,(h\,l) = 0$ when $k > 2h$. With $\eta = h\,l$ it is found that

$$\Sigma\, g_k(\eta) = \tfrac{1}{2} + \sum_1^{h-1} (g_k + g_{2h-k}) = \tfrac{1}{2} + (h-1) = \frac{\eta}{l} - \tfrac{1}{2}.$$

The relation can be expected to hold also for values of η not equal to an integer multiple of l.

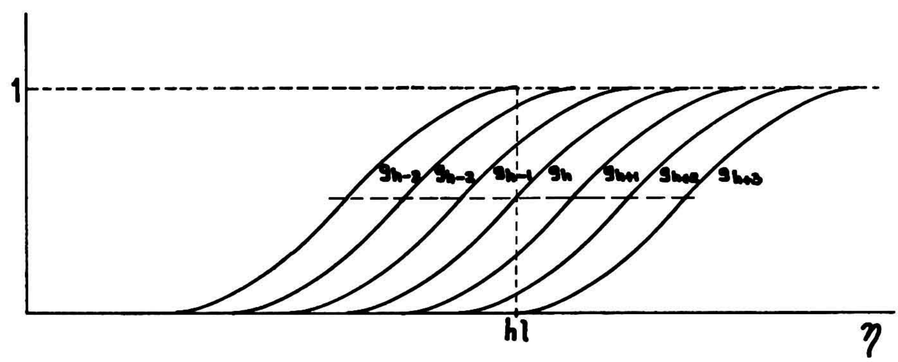

Fig. 8.

Equation (80b) expresses the condition that the dispersion of T'_h about its mean value $h\,l$ shall not increase with h. This follows when we calculate:

$$\overline{(T'_h-hl)^2}=h\,(\omega+2\,\omega_1+2\,\omega_2+2\,\omega_3+\ldots)\,l^2-2\,(\omega_1+2\,\omega_2+3\,\omega_3+\ldots)\,l^2.$$

When (80b) is satisfied, the first sum vanishes, while the latter sum converges to a value independent of k.

As was remarked in **12**, it may be that the substitution of (80a) and (80b) for (79) introduces an unnecessary restriction. What is actually required is that $\overline{(T'_h-\Lambda_h)^2}$ shall not increase with h.

Since the sums considered in eqs. (70) and (76) will occur repeatedly in the following sections, not only for large values of η, but also for other values, it is convenient to shorten notation by writing:

$$(81)\qquad \sum_{k=1}^{\infty} f_k(\eta)\,\overline{\tau_i\tau_{i+k}}^{\,*}=l\cdot\Pi(z)$$

$$(82)\qquad \sum_{k=1}^{\infty} f_k(\eta)\,\overline{\zeta_i\tau_i\tau_{i+k}}^{\,*}=l^2\cdot\Omega(z)$$

where $z=\eta/l$. Equations (70) and (76) then become:

$$(83)\qquad \sum_1^{\infty}\varphi_k=\int_0^z\Pi\,dz$$

$$(84)\qquad \sum_1^{\infty}(\Phi_k-\chi_k)=2\int_0^z\Omega\,dz$$

It follows from (73) that

$$(83a)\qquad \int_0^z\Pi\,dz\cong z-\tfrac{1}{2}(1+\omega)\qquad(\text{for } z\to\infty)$$

and from (77a) that

$$(84a)\qquad \int_0^{\infty}\Omega\,dz=\tilde{\omega}$$

22. *Fourier transform of* $\overline{v_1v_2}$. — According to section 3 the Fourier transform $\Gamma(n)$ is defined by

$$(17a)\qquad \pi\,\Gamma(n)=2\int_0^{\infty}\overline{v_1v_2}\cos n\eta\,d\eta$$

It should be kept in mind that, in $\Gamma(n)$ and in all following equations, n is an arbitrary quantity of dimension (length)$^{-1}$ and not an integer. — We make use of formula (47) for $\overline{v_1v_2}$. The circumstance that it is slightly in error for very small values of η can be left out of account so long as n

is not too large. Integration by parts, having regard to the vanishing of $\overline{v_1 v_2}$ and of its derivative for $\eta \to \infty$, gives

$$\pi\,\Gamma(n) = \frac{\beta^2 l(1+\omega)}{n^2} + \frac{2\beta^2 l}{n^2}\int_0^\infty (\Pi - 1)\cos n l z\,dz \tag{85}$$

The correction which must be applied for very large values of n, can be found if we make use of the expression (49), which gives the correction to $\overline{v_1 v_2}$ connected with the rounding off of the jump at ζ_i. The FOURIER transform of (49) (multiplied by π) is [8]:

$$-\frac{\beta^2 \tau_i^2}{n^2}\left\{1 - \frac{(2\pi n \nu/\beta\tau_i)^2}{[\sinh(2\pi n \nu/\beta\tau_i)]^2}\right\}.$$

When the mean value is taken, the first term cancels the first term of (85), which is replaced by:

$$\overline{\frac{4\pi^2 \nu^2/l}{[\sinh(2\pi n \nu/\beta\tau_i)]^2}} \tag{85a}$$

So long as n is small compared with $\beta\, l/\nu$, formula (85) is sufficiently accurate. Since $\Pi - 1$ is a bounded quantity without discontinuities, the integral in (85) will be of order n^{-1}, making the whole second term of (85) of order n^{-3}. Hence for not too small n we may write:

$$\pi\,\Gamma(n) \cong \frac{l(1+\omega)}{n^2 t^2} \tag{85b}$$

This indicates a spectral function proportional to n^{-2}. The change of $\Gamma(n)$ with time in this range can be found with the aid of (57).

Since $\Gamma(n)$ measures the energy distribution, the amplitude of the spectral components will be proportional to n^{-1}; while for each separate component in this range, the dissipation in unit time will be *independent of* n. This result is analogous to that obtained in previous work, referring to an equation of similar type as eq. (1), but completed with a term representing the action of an outward agency and applied to a limited domain $0 \leqslant y \leqslant b$ [9]). It is a consequence of the nature of the jumps occurring in the function $v(y)$.

The range where (85b) is valid, ends when n approaches $\beta\, l/\nu = l/\nu\, t$. From then onward the spectral function decreases more rapidly than n^{-2}.

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[8]) Compare D. BIERENS DE HAAN, Nouv. Tables d'Intégrales Définies (Leiden 1867), Table 264, no. 2, from which the integral required can be obtained by differentiation with respect to p.

[9]) See: Mathematical examples illustrating relations in the theory of turbulent fluid motion, Verhand. Kon. Nederl. Akademie v. Wetenschappen (1e sect.), vol. XVII, no. 2, 18—33 (1939); or the paper in Advances in Applied Mechanics, 1, 175 (1948).

Let us finally consider very small values of n. In the integral defining $\Gamma(n)$ we develop the cosine function. This will bring us to the development already indicated in formula (18a), with

$$\pi \Gamma_{2m} = 2 \int_0^\infty \overline{v_1 v_2}\, \eta^{2m}\, d\eta$$

(m being an integer), provided we may assume that the integrals converge [10]). By means of partial integration this expression can be transformed into:

$$\pi \Gamma_{2m} = -\frac{2\beta^2 l^{2m+3}}{(2m+1)(2m+2)} \int_0^\infty (\Pi - 1)\, z^{2m+2}\, dz \tag{86}$$

23. *Fourier transform of* $\overline{v_1{}^2 v_2}$ *and application of equation* (19). — We have

$$\pi \Psi(n) = 2 \int_0^\infty \overline{v_1^2 v_2}\, \sin n\eta\, d\eta \tag{17b}$$

With the aid of (53) this can be reduced to:

$$\pi \Psi(n) = -\frac{2\beta}{n}\, \pi \Gamma(n) + \frac{4\beta^3 l^2}{n^2} \int_0^\infty \Omega \sin n l z\, dz \tag{87}$$

When this result is substituted into eq. (19), which is the FOURIER transform of the fundamental equation, the following relation can be obtained:

$$\frac{\partial}{\partial t}\left(\frac{\pi \Gamma}{\beta^2}\right) + 2\nu n^2 \frac{\pi \Gamma}{\beta^2} = \frac{4\beta l^2}{n} \int_0^\infty \Omega \sin n l z\, dz \tag{88}$$

The term with ν apparently can be discarded when $n \ll (\nu t)^{-\frac{1}{2}}$.

When n is small, we develop the sine function in order to arrive at the series (18b) with

$$\pi \Psi_{2m-1} = 2 \int_0^\infty \overline{v_1^2 v_2}\, \eta^{2m-1}\, d\eta,$$

again assuming that the integrals converge. By means of partial integration this can be transformed into:

$$\pi \Psi_{2m-1} = \frac{\beta}{m}\, \pi \Gamma_{2m} - \frac{2\beta^3 l^{2m+3}}{m(2m+1)} \int_0^\infty \Omega\, z^{2m+1}\, dz \tag{89}$$

Equation (20) then leads to:

$$\frac{d}{dt}\left(\frac{\pi \Gamma_{2m}}{\beta^2}\right) - 4\nu m(2m-1) \frac{\pi \Gamma_{2m-2}}{\beta^2} = \frac{4\beta l^{2m+3}}{2m+1} \int_0^\infty \Omega\, z^{2m+1}\, dz \tag{90}$$

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[10]) No difficulty will arise with the integrals appearing in (86), (89) and (90), if we assume that the functions $f_k(\Lambda_k)$ vanish exponentially for $\Lambda_k \to \infty$.

24. *Similarity considerations.* — An important tool in the investigation of turbulence has become the assumption of the preservation of statistical similarity during the development in time [11]). When applied to the system under consideration this assumption requires that, under the laws of motion stated in section 7, mean values of quantities of the same degree in λ_i, τ_i, ζ_i should keep a constant ratio. A consequence is that the dimensionless quantities ω, ω^*, $\tilde{\omega}$ introduced in (31) must remain constant during the development of the system. The same will apply to the ω_k defined in (78); and also to the ω_k^* and similar restricted mean values, provided the value of λ_i or Λ_k to which they refer is made to change proportionally with l.

We give attention in the first place to the expression (35) for the invariant J_0, which we assume to be different from zero. The mean value $\overline{\tau_i \tau_{i+k} \zeta_i \zeta_{i+k}}$ in a self-preserving system must be proportional to l^4. Since J_0 is independent of the time, we thus arrive at the result:

$$\beta^2 l^3 = l^3/t^2 = J_0/c$$

(c being a numerical quantity), from which:

$$l \sim t^{2/3}. \tag{91}$$

It follows immediately that

$$E \sim \overline{v^2} \sim \beta^2 l^2 \sim t^{-2/3} \tag{92a}$$

while:

$$\varepsilon \sim t^{-5/3} \tag{92b}$$

From eq. (9) we then obtain

$$\tfrac{2}{3} E/t = \tfrac{2}{3} \beta E = \varepsilon,$$

and reference to eqs. (32) and (33) leads to the relation:

$$\tilde{\omega} = \tfrac{1}{6} (1 + \omega^*) \tag{93}$$

from which, in connection with (34) [12]):

$$\lambda^2 = 3 \nu t \tag{94}$$

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[11]) Compare papers by TH. DE KARMAN, G. K. BATCHELOR, F. N. FRENKIEL, C. C. LIN and others. We mention in particular: G. K. BATCHELOR, Energy decay and self-preserving correlation functions in isotropic turbulence, Quart. Appl. Mathem. **6**, 97, section 6 (1948).

[12]) It will be recognised that the quantity λ, defined in eqs. (7)—(9) as a kind of minimum length associated with the correlation function essentially depending on the rounding off of the jumps in $v(y)$, which quantity is used again in (34) and here in (94), has nothing to do with the λ applied in eqs. (36) etc. and in eqs. (69), (70) etc. as an integration variable representing the length of a segment λ_i between two jumps (or the sum Λ_k of the lengths of a set of consecutive segments). It is trusted that no difficulties will arise through this double use of λ.

Hence $\lambda \sim t^{1/3}$, which means that l and λ are *not* proportional. Similarity consequently is not possible for all aspects of the system; the type of similarity considered here applies to relations involving dimensions large compared with $\sqrt{\nu t}$. Since $l \sim t^{2/3}$, the difference between l and λ increases with t, so that the range to which this similarity applies is continually extended. It is of interest to note that the REYNOLDS' number Re_I increases proportionally with $t^{1/3}$, since $L \sim l \sim t^{2/3}$ and $v \sim E^{1/2} \sim t^{-1/3}$. — The failure of similarity in the smallest dimensions will be responsible for the difference between the limit found for n in connection with formulae (85*a*), (85*b*), *viz.* $\beta\, l/\nu \sim t^{-1/3}$, and the limit found in connection with (88), *viz.* $(\nu\, t)^{-1/2}$.

Equation (65) giving dl/dt now leads to the important formula

$$\tag{95} \tfrac{1}{2} f_0 \, \overline{(\tau_i + \tau_{i+1})}^{\,*} = f_0 \, \overline{\tau_i}^{\,*} = \tfrac{2}{3}$$

When this is substituted into (68) we are brought back to (93).

Another interesting result is obtained from (66):

$$\tag{96} f_0 \, \overline{(\tau_i + \tau_{i+1})\, \tau_i \, \tau_{i+1}}^{\,*} = \tfrac{2}{3}\,(1+\omega)\, l^2$$

Turning to the spectrum we observe that $\Gamma_0 = 2\, J_0/\pi$ is independent of the time. Hence we must expect that the same will apply to $\Gamma(n)$, provided we make n vary inversely proportionally to l, *i.e.* $n \sim t^{-2/3}$. Indeed it will be seen from (85*b*) that this makes $\Gamma(n)$ constant. Apparently the head of the spectrum contracts towards the lower wave numbers. This is a consequence of the reduction of the number of segments λ_i and τ_i, considered in section 7.

25. The similarity hypothesis makes it possible to write $\overline{v_1 v_2}$ in the form:

$$\tag{a} \overline{v_1\, v_2} = \beta^2\, l^2\, \psi\,(\eta/l)\,,$$

where ψ does not contain the time in an explicit way. We then find

$$\tag{b} \frac{\partial}{\partial t}\, \overline{v_1\, v_2} = -\tfrac{2}{3}\, \beta^3\, l^2 \left(\psi + \frac{\eta}{l}\, \psi'\right).$$

When this result is substituted into the fundamental equation (12*a*) and use is made of (47) and (54), the following formula is obtained:

$$\sum_1^\infty \left(2\, \frac{\eta}{l}\, \varphi_k - 3\, \Phi_k - \chi_k\right) = -(1+\omega)\, \frac{\eta}{l} - 2\, \tilde{\omega} + \tfrac{1}{3}\,(1+\omega^*)\,,$$

which is not confined to large values of η, but must be valid for all values of η (except those of order $\nu\, t/l$).

Substitution of $\eta = 0$ leads us back to (93), so that the equation is also valid here. We can therefore omit the last two terms on the right hand side, which leaves us with:

$$\sum_1^\infty \left(2\frac{\eta}{l}\varphi_k - 3\,\Phi_k - \chi_k\right) = -(1+\omega)\frac{\eta}{l} \tag{97}$$

When we divide by η/l and go to the limit $\eta = 0$, we come back to (96). — With the aid of the results of section **20** it is easy to show that the equation is satisfied for $\eta \to \infty$.

When (97) is differentiated with respect to η, and attention is given to (81), (82) and (84), the following equation can be derived:

$$\Omega = \tfrac{1}{6}\,(1+\omega) - \tfrac{1}{3}\,z^2 \frac{d}{dz}\left(\frac{1}{z}\int_0^z \Pi\, dz\right) \tag{98}$$

It can be surmised that this relation between the functions Ω and Π, obtained for self-preserving systems, will play an important part in the statistical relations characterising such systems.

For $z = 0$ the equation gives $\Omega = \frac{1}{6}(1+\omega)$. The same value is obtained from (82) if it is observed that all $f_k(\eta)$ go to zero for $\eta = 0$, with the exception of $f_1(\eta)$, which has the limiting value f_0. Since $\zeta_i - \zeta_{i+1} = \frac{1}{2}(\tau_i + \tau_{i+1})$ when $\lambda_{i+1} = 0$, the only term remaining in (82) can be readily reduced with the aid of (96).

It is further found that $d\,\Omega/d\,z = 0$ for $z = 0$.

For large values of z eq. (83*a*) gives $\int \Pi\, d\,z = z - \frac{1}{2}(1+\omega)$, which duly leads to $\Omega = 0$ for $z \to \infty$.

Equation (98) can also be solved for Π. Introducing the result into (83) we obtain:

$$\sum_1^\infty \varphi_k = C\,z - \tfrac{1}{2}\,(1+\omega) + 3\,z\int_z^\infty \frac{\Omega}{z^2}\,dz \tag{99}$$

This is an extension of (73), now valid for all values of η. The coefficient C (which makes its appearance as an integration constant) evidently must be equal to 1.

Formula (48) now gives:

$$\frac{\partial}{\partial\eta}\,\overline{v_1\,v_2} = -\,3\,\beta^2 l\,z\int_z^\infty \frac{\Omega}{z^2}\,dz\,,$$

from which by integration:

$$\overline{v_1\,v_2} = \tfrac{3}{2}\,\beta^2\,l^2\left(\int_z^\infty \Omega\,dz - z^2\int_z^\infty \frac{\Omega}{z^2}\,dz\right) \tag{100}$$

Since we have seen that Ω has a finite value for $z=0$, the second term between the brackets in (100) vanishes for $z=0$ and we obtain:

(101) $$\overline{v^2} = \tfrac{3}{2}\beta^2 l^2 \int_0^\infty \Omega\, dz$$

from which, in connection with (32) and (93):

(102) $$\int_0^\infty \Omega\, dz = \tilde{\omega} = \tfrac{1}{6}(1+\omega^*)$$

We can also form:

$$\sum_1^\infty \chi_k = \int_0^z \Pi z\, dz = \tfrac{1}{2} z^2 - \tfrac{3}{2}\int_0^z \Omega\, dz + \tfrac{3}{2} z^2 \int_z^\infty \frac{\Omega}{z^2}\, dz$$

$$\sum_1^\infty \Phi_k = \int_0^z (\Pi z + 2\Omega)\, dz = \tfrac{1}{2} z^2 + \tfrac{1}{2}\int_0^z \Omega\, dz + \tfrac{3}{2} z^2 \int_z^\infty \frac{\Omega}{z^2}\, dz$$

$$\sum_1^\infty X_k = \int_0^z (\Pi z^2 + 2\Omega z)\, dz = \tfrac{1}{3} z^3 + z^3 \int_z^\infty \frac{\Omega}{z^2}\, dz$$

$$\frac{d}{d\eta}\overline{(v_1^2 v_2)} = -\beta^3 l^2 \left(\int_z^\infty \Omega\, dz - 3 z^2 \int_z^\infty \frac{\Omega}{z^2}\, dz\right)$$

(103) $$\overline{v_1^2 v_2} = -\beta^3 l^3 \left(z \int_z^\infty \Omega\, dz - z^3 \int_z^\infty \frac{\Omega}{z^2}\, dz\right)$$

From the latter expression it follows that for self-preserving systems

(104) $$\overline{v_1^2 v_2} = -\tfrac{2}{3}\beta\eta\, \overline{v_1 v_2}$$

It will be seen that this relation can be obtained immediately from eqs. (*a*) and (*b*) at the beginning of this section, since these give

$$\frac{\partial}{\partial t}\overline{v_1 v_2} = -\tfrac{2}{3}\beta \frac{\partial}{\partial \eta}(\eta \cdot \overline{v_1 v_2})$$

for self-preserving systems. Substitution into (12*a*) and integration immediately leads to (104).

Finally, after some transformations, we find:

(105) $$\pi\, \Gamma(n) = \frac{6\beta^2 l^2}{n^3} \int_0^n dn\, n \int_0^\infty \Omega \sin nlz\, dz$$

By calculating the limit for $n \to 0$ we obtain

(106) $$J_0 = \tfrac{1}{2}\pi\, \Gamma_0 = \beta^2 l^3 \int_0^\infty \Omega z\, dz$$

which can also be deduced directly from (100). If we may assume that $d\,\Omega/d\,z \leqslant 0$ for all z, we can calculate the limiting form for large n, which gives:

$$\pi\,\Gamma(n) = \frac{6\,\beta^2 l}{n^2}\,\Omega(0) = \frac{\beta^2 l\,(1+\omega)}{n^2}.$$

This is in accordance with (85*b*).

26. We add an observation concerning J_0. Introducing the "momenta"

$$\mu_i = \beta\,\tau_i\,\zeta_i \tag{107}$$

[(compare footnote 3) to section 7)], we can write:

$$J_0 = \frac{1}{l}\left(\tfrac{1}{2}\,\overline{\mu_i^2} + \sum_{k=1}^{\infty} \overline{\mu_i\,\mu_{i+k}}\right) \tag{108}$$

We have already seen that the μ_i do not change during the normal motion. When two vertical segments τ_i, τ_{i+1} combine to form a single segment, the corresponding momenta μ_i, μ_{i+1} simply add.

We consider the change in the course of time suffered by the various terms of the sum (108) in consequence of the combination of segments. For shortness we write:

$$a = \tfrac{1}{2}\,f_0\,(\tau_i + \tau_{i+1})/t \qquad (\text{for } \lambda_i \to 0).$$

From a consideration of terms disappearing and newly appearing in the sums, upon coalescence of segments, it is found that:

$$\frac{d}{dt}\left(\tfrac{1}{2}\,N\,\overline{\mu_i^2}\right) = N\,\overline{a\,\mu_i\,\mu_{i+1}}^{\,*}$$

$$\frac{d}{dt}\,N\,\overline{\mu_i\;\mu_{i+1}} = 2N\,\overline{a\,\mu_i\,\mu_{i+2}} - N\,\overline{a\,\mu_i\,\mu_{i+1}}^{\,*}$$

$$\frac{d}{dt}\,N\,\overline{\mu_i\;\mu_{i+k}} = (k+1)\,N\,\overline{a\,\mu_i\,\mu_{i+k+1}} - k\,N\,\overline{a\,\mu_i\,\mu_{i+k}}\,.$$

In the first two equations the asterisk indicates the restricted mean value for $\lambda_i \to 0$. In the other terms we have omitted the asterisk, since Λ_k $(k > 1)$ does not become zero; it is assumed that ordinary mean values could be used. Whether this may be correct or an approximation only, it will be seen that for every case of coalescence the terms appearing in one mean value, disappear from the next higher one. Hence it is found again that

$$d\,J_0/dt = 0.$$

However, if the similarity hypothesis holds, every separate mean value must satisfy the relation

$$\overline{\mu_i\,\mu_{i+k}} \backsim \beta^2\,l^4\,,$$

so that

$$\frac{d}{dt} N \overline{\mu_i \, \mu_{i+k}} \sim \frac{d}{dt} \beta^2 l^3 = 0 \, .$$

This requires that $k \, \overline{a \, \mu_i \, \mu_{i+k}}$ shall be independent of k. We cannot suppose that this should be a constant differing from zero, for this would make the sum divergent. We are thus led to the conclusion that these quantities must be zero for $k \neq 0$. This will be the case if it can be assumed that all ζ_i are independent of each other, so that

$$\overline{\zeta_i \, \zeta_{i+k}} = 0 \quad (k \neq 0) \tag{109}$$

and likewise $\overline{\mu_i \, \mu_{i+k}} = 0$ for all k different from zero. The expression for J_0 then becomes:

$$J_0 = \frac{\beta^2}{2l} \overline{\tau_i^2 \zeta_i^2} = \frac{\overline{\mu_i^2}}{2l} \tag{110}$$

It is probable that in sections **24** — **26** we have obtained the principal relations that can be deduced from the hypothesis of similarity. The results of section **25** have shown that the relevant statistical quantities for self-preserving systems depend upon a single function, for which we may take either $\Pi(z)$ or $\Omega(z)$. The form of this function, however, remained unknown and it is not to be expected that the similarity hypothesis can help us much further in this respect. Substitution of the full expressions (47) and (54) for $\overline{v_1 v_2}$ and $\partial(\overline{v_1^2 v_2})/\partial\eta$ into the fundamental equation (12*a*) can give a lot of relations we have not made use of, but it has become evident that every new equation brings new statistical functions, so that it looks as if there will always be more unknowns than equations.

The only way to obtain further information will be an attack upon the intrinsic statistical problem. Although the complicated relations between the λ_i, τ_i, ζ_i make a solution of this problem beyond our power, it will be attempted to discuss certain aspects in the last part of this paper.

(To be continued).