MATHEMATICS

A SYMMETRIC FORM OF GÖDEL'S THEOREM *)

BY

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It has been remarked, particularly in articles of Mostowski [1]), that recursively enumerable sets behave surprisingly similarly to analytic sets and general recursive sets to Borel sets. It is a theorem of Lusin that two disjoint analytic sets can always be separated by a Borel set, i.e. this Borel set contains one of the given analytic sets and is disjoint from the other [2]). We shall construct two disjoint recursively enumerable sets C_0 and C_1 which cannot be separated by a general recursive set. This example shows that there is no exact parallelism between the two theories [3]).

We actually establish the following property of the sets C_0 and C_1, which is stronger constructively: Given any two disjoint recursively enumerable sets D_0 and D_1 such that $C_0 \subset D_0$ and $C_1 \subset D_1$, there can always be found a number f such that $\overline{f \,\varepsilon\, D_0 + D_1}$.

Let T_1 be the primitive recursive predicate so designated in a previous paper by the author [4]), and let $(x)_i$ be the number of times x contains the $i+1$-st prime number as factor (0, if $x = 0$) [5]). Let predicates W_0

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[1]) Andrzej Mostowski, *On definable sets of positive integers*, Fundamenta Mathematicae, 34, 81—112 (1946), and *On a set of integers not definable by means of one-quantifier predicates*, Annales de la société polonaise de mathématique, 21, 114—119 (1948).

[2]) See p. 52 of N. Lusin, *Sur les ensembles analytiques*, Fundamenta mathematicae, 10, 1—95 (1927); or Casimir Kuratowski, *Topologie I*, Monografie Matematyczne, Warsaw-Lwów 249 (1933).

[3]) The example does not go against the parallelism between the theory of recursive predicates and quantifiers and the corresponding theory formulated by Mostowski 1946 [1]) in terms similar to the theory of projective sets. In § 6 of Mostowski's paper it is shown that these theories are equivalent, unless we admit as the basic system S for his theory one which does not satisfy two recursivity conditions (R_1) and (R_2). All ordinary (constructive) formal systems for arithmetic satisfy these conditions.

[4]) S. C. Kleene, *Recursive predicates and quantifiers*, Transactions of the American Mathematical Society, 53, 41—73 (1943).

[5]) This $(x)_i$ is a primitive recursive function of x and i; in the notation of S. C. Kleene, *General recursive functions of natural numbers*, Mathematische Annalen, 112, 727—742 (1936), no. 6, p. 732, $(x)_i = i + 1\ Gl\ x$.

and W_1 be defined thus,

$$W_0(x, y) \equiv T_1((x)_1, x, y) \ \& \ (z)\{z \leqslant y \rightarrow \overline{T}_1((x)_0, x, z)\},$$
$$W_1(x, y) \equiv T_1((x)_0, x, y) \ \& \ (z)\{z \leqslant y \rightarrow \overline{T}_1((x)_1, x, z)\},$$

and the sets C_0 and C_1 as follows,

$$C_0 = \hat{x}\,(Ey)\,W_0(x, y), \qquad C_1 = \hat{x}\,(Ey)\,W_1(x, y).$$

The predicates W_0 and W_1 are primitive recursive [6]); hence the sets C_0 and C_1 are recursively enumerable [7]). From $W_0(x, y_0)$ and $W_1(x, y_1)$ we can infer both $y_0 > y_1$ and $y_1 > y_0$; hence

(1) $$\overline{(Ey)\,W_0(x, y) \ \& \ (Ey)\,W_1(x, y)},$$

i.e. C_0 and C_1 are disjoint.

Consider any two disjoint recursively enumerable sets D_0 and D_1 such that $C_0 \subset D_0$ and $C_1 \subset D_1$. We can write $D_0 = \hat{x}\,(Ey)\,R_0(x, y)$ and $D_1 = \hat{x}\,(Ey)\,R_1(x, y)$ with R_0 and R_1 recursive.

Now we show that there is a number f such that $\overline{f \,\varepsilon\, D_0 + D_1}$.

By the enumeration theorem for predicates of the form $(Ey)\,R(x, y)$ with R recursive [8]), there are numbers f_0 and f_1 such that, if we put $f = 2^{f_0} \cdot 3^{f_1}$, then

(2) $$(Ey)\,R_0(x, y) \equiv (Ey)\,T_1(f_0, x, y) \equiv (Ey)\,T_1((f)_0, x, y),$$

(3) $$(Ey)\,R_1(x, y) \equiv (Ey)\,T_1(f_1, x, y) \equiv (Ey)\,T_1((f)_1, x, y).$$

Assume: (a) $f \,\varepsilon\, D_0$, i.e. $(Ey)\,R_0(f, y)$. Then by (2): (b) $(Ey)\,T_1((f)_0, f, y)$. Also by (a) and the disjointness of D_0 and D_1: (c) $\overline{f \,\varepsilon\, D_1}$, i.e. $(\overline{Ey})\,R_1(f, y)$. Thence by (3), $(\overline{Ey})\,T_1((f)_1, f, y)$; whence: (d) $(y)\,\overline{T}_1((f)_1, f, y)$. By (b) and (d), $(Ey)\,[T_1((f)_0, f, y) \ \& \ (z)\{z \leqslant y \rightarrow \overline{T}_1((f)_1, f, z)\}]$, i.e. $(Ey)\,W_1(f, y)$, i.e. $f \,\varepsilon\, C_1$. Since $C_1 \subset D_1$, therefore $f \,\varepsilon\, D_1$, contradicting (c). By reductio ad absurdum, therefore (a) is false; i.e.

(4) $$\overline{f \,\varepsilon\, D_0}.$$

By a similar argument, or thence by the symmetry of the conditions on C_0 and D_0 to those on C_1 and D_1,

(5) $$\overline{f \,\varepsilon\, D_1}.$$

Thus there is no separation of all natural numbers into two disjoint recursively enumerable sets D_0 and D_1 such that $C_0 \subset D_0$ and $C_1 \subset D_1$.

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[6]) See e.g. Kurt Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I*, Monatshefte für Mathematik und Physik, **38**, 173—198 (1931), Theorems II and IV.

[7]) Kleene 1936 [5]) Theorem III. Members of the sets C_0 and C_1 are easily found; e.g. if we take $x = x \ \& \ y = y$ as the $R(x, y)$ in Kleene 1943 [4]) Theorem I, then $(Ey)\,W_0(2^g \cdot 3^f, y)$ and $(Ey)\,W_1(2^f \cdot 3^g, y)$.

[8]) Kleene 1943 [4]) Theorem I.

This of course implies, and by the theorem for recursive predicates and quantifiers [9]) analogous to SOUSLIN's theorem for analytic and Borel sets [10]) is actually equivalent to, the statement that C_0 and C_1 cannot be separated by any general recursive set.

The root of this example is ROSSER's method [11]) of weakening the hypothesis of ω-consistency to simple consistency for GÖDEL's proof of the existence of an undecidable proposition in a formal system containing arithmetic [12]). The author mentioned previously that ROSSER's form of GÖDEL's theorem (as well as the original form) can be brought under a general theorem on recursive predicates and quantifiers [13]). The present result is obtained by rearranging the argument to make it symmetrical between the proposition and its negation. A discussion of it from this standpoint is included in another manuscript by the author. Upon seeing that manuscript, MOSTOWSKI pointed out the contrast to a theorem holding for analytic and Borel sets.

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[9]) KLEENE 1943 [4]) Theorem V, or p. 290 of EMIL L. POST, *Recursively enumerable sets of positive integers and their decision problems*, Bulletin of the American Mathematical Society, **50**, 284—316 (1944), or MOSTOWSKI 1946 [1]) 5.51. The present application is valid intuitionistically.

[10]) M. SOUSLIN, *Sur une définition des ensembles mesurables B sans nombres transfinis*, Comptes Rendus hebdomadaires des séances de l'Academie des Sciences, Paris, **164**, 88—91 (1917), Theorem III; KURATOWSKI 1933 [2]) p. 251 Corollary 1.

[11]) BARKLEY ROSSER, *Extensions of some theorems of* GÖDEL *and* CHURCH, The Journal of Symbolic Logic, **1**, 87—91 (1936), Theorem II.

[12]) GÖDEL 1931 [6]) Theorem VI.

[13]) KLEENE 1943 [4]) p. 64.