ON SOME VOLTERRA INTEGRAL EQUATIONS OF WHICH ALL SOLUTIONS ARE CONVERGENT

BY

N. G. DE BRUIJN

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1. *Introduction.*

We shall be concerned with equations of the type

$$f(x) = \int_0^1 K(x,t)\, f(x-t)\, dt. \tag{1. 1}$$

The problem is to impose conditions on K which guarantee that every solution $f(x)$ will be convergent, i.e. $\lim_{x\to\infty} f(x)$ exists. Throughout the paper we suppose that the kernel is absolutely integrable with respect to x and t for $0 \leqslant x \leqslant a$, $0 \leqslant t \leqslant 1$ and any $a > 0$. Furthermore we shall assume that, for $x \geqslant 1$,

$$K(x,t) \geqslant 0 \ (0 \leqslant t \leqslant 1); \ K(x,t) = 0 \ (t > 1 \text{ and } t < 0); \ \int_0^1 K(x,t)\, dt = 1. \tag{1. 2}$$

A function $f(x)$ will be called a *solution*, if it is measurable and bounded on any finite interval $0 \leqslant x \leqslant a$, and if it satisfies (1. 1) for $x \geqslant 1$. Since we do not assume $K(x,t)$ to be uniformly bounded, a special difficulty arises, which is demonstrated by the following example. Take

$$\begin{cases} K(x,t) = 1 \text{ if } x \text{ is an integer} \\ K(x,t) = (x-[x])^{-1} \text{ for } 0 \leqslant t \leqslant x-[x] \text{ if } x \text{ is not an integer} \\ K(x,t) = 0 \text{ otherwise.} \end{cases} \tag{1. 3}$$

Now solutions $f(x)$ can be constructed from arbitrary sequences $c_1, c_2, c_3, \ldots$ of real numbers by taking $f(x) = c_n$ for $n-1 < x \leqslant n$. In this example $f(x)$ is not uniquely determined by its values on the interval $0 \leqslant x \leqslant 1$. Nevertheless the kernel (1. 3) has a stabilizing effect: every *continuous* solution is constant for $x \geqslant 1$.

In order to avoid the difficulty shown by this example, we shall consider *regular* solutions only. A solution $f(x)$ is called regular, if, for all values of m, M and $a \geqslant 1$ the following statement holds true: *If $m \leqslant f(x) \leqslant M$ almost everywhere on $a-1 \leqslant x \leqslant a$, then we have $m \leqslant f(x) \leqslant M$ for $x > a$.* It follows that a regular solution is uniquely determined by its values on $0 \leqslant x \leqslant 1$.

The problem as to which kernels have the property that any solution is regular, will not be considered in this paper. Neither shall we deal with the question whether any function $f(x)$ given on $0 \leqslant x \leqslant 1$ can be continued to a solution [1]).

In practice the two following theorems are often sufficient.

Theorem 1. *Every continuous solution is regular.*

Proof. Let $f(x)$ be continuous, and assume $f(x) \leqslant M$ for $a-1 \leqslant x \leqslant a$. Assume that $x_1 > a$ and $f(x_1) > M$. Let x_2 be the least possible number such that $a < x_2 \leqslant x_1$, $f(x_2) = f(x_1)$. Now we have $f(x) < f(x_2)$ for $x_2 - 1 \leqslant x < x_2$. Apply (1. 1) with $x = x_2$, and a contradiction follows.

Theorem 2. *If, for any finite $A > 0$, the kernel $K(x, t)$ is bounded over $1 \leqslant x \leqslant A$, $0 \leqslant t \leqslant 1$, then every solution is regular.*

Proof. We can find $\varepsilon > 0$ (ε depending on A), such that $\int\limits_0^\varepsilon K(x,t)\,dt < \frac{1}{2}$ for $1 \leqslant x \leqslant A$. Now assume $a > 1$, $A > a$, and $f(x) \leqslant M$ for $x < a$. By the definition of a solution, $f(x)$ is bounded for $a \leqslant x \leqslant a + \varepsilon$; denote its upper bound by M_1. Assume $M_1 > M$. By (1. 1) we find, for $a \leqslant x \leqslant$ $\leqslant a + \varepsilon$, that $f(x) \leqslant M + \frac{1}{2}(M_1 - M)$, which means that the upper bound for $a \leqslant x \leqslant a + \varepsilon$ would be less than M_1. This being contradictory, we infer $M_1 \leqslant M$. The theorem now follows by induction.

A kernel $K(x, t)$ will be called *stabilizing*, if every regular solution is convergent. In sections 2 and 3 we shall give sufficient conditions for a kernel to be stabilizing.

In section 4 we give two examples; of which the first one is the case where $K(x, t)$ does not depend on x.

We here refer to the possibility of extending theorem 3 (section 2) to the case of an equation involving a Stieltjes integral, viz.

$$f(x) = \int\limits_0^1 f(x-t)\, dW_x(t).$$

Here, for any $x \geqslant 1$, $W_x(t)$ has to be increasing, and $W_x(0) = 0$, $W_x(1) = 1$, and we content ourselves with the consideration of the behaviour of continuous solutions $f(x)$. Only formal changes in the statement and proof of Theorem 3 are necessary. Again, every continuous solution is regular, provided that for no value of x we have

$$W_x(t) = 0 \qquad (0 \leqslant t < 1), \quad W_x(1) = 1.$$

2. *Sufficient condition for a kernel to be stabilizing.*

In order to show how far we can certainly *not* go, we give two examples of kernels with a regular but non-convergent solution.

A. If we take $f(x) = \dfrac{x}{x+1} \sin 2\pi x$, we can construct, in very many ways, kernels admitting $f(x)$ as a solution. But if x aproaches an integer,

[1]) If K is uniformly bounded, these problems are relatively simple. See Volterra, Leçons sur les equations intégrales, (Paris 1913).

we cannot avoid concentrating the power of $K(x, t)$, considered as a function of t, near the points $t = 0$ and $t = 1$.

Indeed, the conditions of both theorem 3 and theorem 4 will exclude too heavy concentrations of power at a finite number of points, for too many values of x.

B. In the following example $K(x, t)$ is uniformly bounded, but not stabilizing. Let N be a natural number. Put $f(x) = 0$ if $0 \leqslant Nx - [Nx] < \frac{1}{2}$, $f(x) = 1$ if $\frac{1}{2} \leqslant Nx - [Nx] < 1$. Take $K(x,t) = 2$ if $x \geqslant 1$, $0 \leqslant t \leqslant 1$ and $f(x) = f(x-t)$; $K(x,t) = 0$ otherwise. Clearly K satisfies (1. 2), and $f(x)$ is a regular solution which does not converge.

This example shows that, in the following theorem, the constant γ may not be taken $\geqslant \frac{1}{2}$.

Theorem 3. *Let γ be a positive constant $< \frac{1}{4}$, and let, for $x \geqslant 1$, $\Phi(x)$ be a continuous function satisfying*

$$\Phi(x) \geqslant 0 \quad (x \geqslant 1), \quad \sum_{n=1}^{\infty} \eta_n = \infty \tag{2. 1}$$

where

$$\eta_n = \min_{n \leqslant x \leqslant n+2} \Phi(x).$$

Now a sufficient condition for $K(x, t)$ (satisfying (1. 2)) to be stabilizing, is that

$$\int_E K(x,t)\, dt \geqslant \Phi(x) \tag{2. 2}$$

for any $x \geqslant 1$, and for any measurable subset E of the interval $0 \leqslant t \leqslant 1$ whose measure is $\geqslant \gamma$.

Proof. Let $f(x)$ be a regular solution. For $x \geqslant 1$, we denote by $M(x)$ and $m(x)$, respectively, the effective maximum and minimum of $f(u)$ for $x - 1 \leqslant u \leqslant x$. It immediately follows from the definition of regularity, that $M(x)$ is non-increasing, and $m(x)$ non-decreasing.

The difference

$$\Delta(x) = M(x) - m(x)$$

is also non-increasing. Putting

$$\int_{x-1}^{x} f(u)\, du = \mathfrak{M}(x)$$

we have, for $x > 1$ almost everywhere

$$\left| \frac{d}{dx} \mathfrak{M}(x) \right| \leqslant \Delta(x) \tag{2. 3}$$

Lemma 1. *If n is a positive integer, we can find a number y in the interval $n + 1 \leqslant y \leqslant n + 2$ such that either*

$$\mathfrak{M}(x) \leqslant M(n) - \tfrac{1}{4}\Delta(n) \qquad \textit{for all } x \textit{ in } y-1 \leqslant x \leqslant y, \tag{2. 4}$$

or

$$\mathfrak{M}(x) \geqslant m(n) + \tfrac{1}{4}\Delta(n) \qquad \textit{for all } x \textit{ in } y-1 \leqslant x \leqslant y. \tag{2. 5}$$

Proof. Assume (2. 4) to be false for all y in $n+1 \leqslant y \leqslant n+2$. Then it is false for $y = n + \frac{3}{2}$, and we can find a number x_0 $(n + \frac{1}{2} \leqslant \leqslant x_0 \leqslant n + \frac{3}{2})$, such that $\mathfrak{M}(x_0) > M(n) - \frac{1}{4}\triangle(n)$.

Now by (2. 3) we have, for $x_0 - \frac{1}{2} \leqslant x \leqslant x_0 + \frac{1}{2}$,

$$\mathfrak{M}(x) > M(n) - \tfrac{1}{4}\triangle(n) - \tfrac{1}{2}\triangle(n) = m(n) + \tfrac{1}{4}\triangle(n).$$

Hence we can take $y = x_0 + \frac{1}{2}$ in (2. 5), which proves the lemma.

Lemma 2. *Let, for a fixed value of* x, $\lambda_1 \geqslant 0$ *be such that, for any subset* E *of* $0 \leqslant t \leqslant 1$ *of measure* $\mu(E) = \dfrac{M(x) - \mathfrak{M}(x)}{M(x) - m(x)}$ *we have* $\int\limits_E K(x, t)\, dt \geqslant \lambda_1$. *Let* $\lambda_2 \geqslant 0$ *be such that for any subset* E' *of measure* $\mu_1(E') = \dfrac{\mathfrak{M}(x) - m(x)}{M(x) - m(x)}$ *we have* $\int\limits_{E'} K(x, t)\, dt \geqslant \lambda_2$. *Then we have*

$$f(x) \leqslant M(x) - \lambda_1 \{M(x) - m(x)\} \tag{2. 6^a}$$

$$f(x) \geqslant m(x) + \lambda_2 \{M(x) - m(x)\}. \tag{2. 6^b}$$

Proof. We only show the truth of (2. 6^a); the other part is analogous. Choose E such that it has the prescribed measure, and such that there is a number p with $K(x, t) \leqslant p$ for $t \in E$, $K(x, t) > p$ for t not on E. Let $\overline{E}$ be the complement of E. Now we have, by (1. 1),

$$f(x) = \int\limits_E K(x, t) f(x-t)\, dt + \int\limits_{\overline{E}} K(x, t) f(x-t)\, dt.$$

Furthermore

$$\int\limits_{\overline{E}} K(x, t)\{M(x) - f(x-t)\}\, dt \geqslant p \int\limits_{\overline{E}} \{M(x) - f(x-t)\}\, dt,$$

$$\int\limits_E K(x, t)\{m(x) - f(x-t)\}\, dt \geqslant p \int\limits_E \{m(x) - f(x-t)\}\, dt.$$

The sum of the right-hand-sides is zero, due to the choice of $\mu(E)$. Hence we obtain

$$f(x) \leqslant m(x) \int\limits_E K\, dt + M(x) \int\limits_{\overline{E}} K\, dt = M(x) - \{M(x) - m(x)\} \int\limits_E K\, dt,$$

and the assertion follows.

We now conclude the proof of theorem 3. In accordance with lemma 1, first assume that there exists an y $(n + 1 \leqslant y \leqslant n + 2)$ such that (2. 4) holds. Now apply (2. 6^a) with $\lambda_1 = \eta_n$ for all x in $y - 1 \leqslant x \leqslant y$. We obtain that either for all x in $y - 1 \leqslant x \leqslant y$

$$f(x) \leqslant M(x) - \eta_n \{M(x) - m(x)\} \tag{2. 7}$$

or for some x_1 $(y - 1 \leqslant x_1 \leqslant y)$

$$\frac{M(x_1) - \mathfrak{M}(x_1)}{M(x_1) - m(x_1)} < \gamma. \tag{2. 8}$$

From (2. 7) we infer

$$M(y) \leqslant (1 - \eta_n) M(y - 1) + \eta_n m(y),$$

and so

$$M(n+2) \leqslant (1-\eta_n)\, M(n) + \eta_n\, m(n+2).$$

Adding the non-negative number $(1-\eta_n)\,\{m(n+2)-m(n)\}$ to the right-hand-side, we obtain

$$\triangle(n+2) \leqslant (1-\eta_n)\,\triangle(n). \tag{2. 9}$$

Now assume (2. 8) to be true. According to our previous assumption, (2. 4) holds for $x = x_1$; combining this with (2. 8) we obtain

$$M(x_1) - \gamma\,\{M(x_1) - m(x_1)\} \leqslant M(n) - \tfrac{1}{4}\,\triangle(n)$$

and hence, by $n \leqslant x_1 \leqslant n+2$,

$$(1-\gamma)\, M(n+2) + \gamma\, m(x_1) - m(n) \leqslant \tfrac{3}{4}\,\triangle(n). \tag{2. 10}$$

And, since $m(n) \leqslant m(x_1) \leqslant m(n+2)$, we have

$$\gamma\, m(x_1) - m(n) = \gamma\,\{m(x_1) - m(n)\} - (1-\gamma)\, m(n) \geqslant -(1-\gamma)\, m(n+2).$$

It now follows from (2. 10) that

$$\triangle(n+2) \leqslant \frac{3}{4(1-\gamma)}\,\triangle(n). \tag{2. 11}$$

Summarizing, the assumption that (2. 4) is true for some y $(n+1 \leqslant \leqslant y \leqslant n+2)$ leads either to (2. 9) or to (2. 11). The same thing can be proved by assuming (2. 5) to be true for some y $(n+1 \leqslant y \leqslant n+2)$. The proof is analogous; we have to use (2. 6^b) instead of (2. 6^a).

Hence, for $n = 1, 2, 3, \ldots$

$$\frac{\triangle(n+2)}{\triangle(n)} \leqslant \operatorname{Max}\left\{\frac{3}{4(1-\gamma)}\,,\; 1-\eta_n\right\}. \tag{2. 12}$$

By (2. 1) we have

$$\prod_{k=1}^{\infty} \operatorname{Max}\left\{\frac{3}{4(1-\gamma)}\,,\; 1-\eta_k\right\} = 0.$$

Hence $\triangle_n \to 0$, and so $M(x)$ and $m(x)$ tend to the same limit, as $x \to \infty$. Since $f(x)$ is a regular solution, we have $m(x) \leqslant f(u) \leqslant M(x)$ for $u \geqslant x$ (without exception for a nul-set.) Consequently $\lim\limits_{x\to\infty} f(x)$ exists, and the theorem is proved.

3. *Results obtained by studying iterated kernels.*

If $K(x,t)$ is very small on sets of measure $\geqslant \frac{1}{4}$, theorem 3 does not apply. Nevertheless we often can prove $K(x,t)$ to be stabilizing by use of iterated kernels.

These iterated kernels are not normalised by the condition

$$\int_0^1 K(x,t)\,dt = 1, \qquad K(x,t) = 0 \;(t > 1).$$

Therefore we use the letters Q, R, S, T, U for such kernels; the letter K is reserved for normalised kernels.

Putting $Q^{(1)}(x, t) = K(x, t)$ $(0 \leqslant t \leqslant 1)$, we define $Q^{(\nu)}(x, t)$ by

$$Q^{(\nu)}(x,t) = 0 \text{ for } t > \nu \qquad (\nu = 1, 2, 3, \ldots) \text{ and for } t < 0,$$

$$Q^{(\nu+1)}(x,t) = \int_0^\infty Q^{(\nu)}(x,s)\, Q^{(1)}(x-s, t-s)\, ds. \tag{3. 1}$$

Then $Q^{(\nu)}(x, t)$ exists almost everywhere, and

$$Q^{(\nu)}(x,t) \geqslant 0 \quad , \quad \int_0^\nu Q^{(\nu)}(x,t)\, dt = 1.$$

Let $f(x)$ be a regular solution of (1. 1), then we have

$$f(x) = \int_0^\nu Q^{(\nu)}(x,t)\, f(x-t)\, dt \qquad (\nu = 1, 2, 3, \ldots ;\ x \geqslant \nu). \tag{3. 2}$$

Now, if m is a natural number, we take

$$K_m(x,t) = \sum_{\nu=1}^{m} Q^{(\nu)}(mx, mt) \quad , \quad f(mx) = g(x) \tag{3. 3}$$

and we obtain

$$g(x) = \int_0^1 K_m(x,t)\, g(x-t)\, dt \qquad (x \geqslant 1). \tag{3. 4}$$

It is easily seen that $K_m(x,t)$ satisfies (1. 2) and that $g(x)$ is a regular solution of (3. 4). Hence $K(x, t)$ is stabilizing whenever $K_m(x, t)$ is stabilizing.

There are cases where theorem 3 applies to K_m but not to K.

We have also the possibility of incomplete iteration, which is demonstrated in the proof of theorem 4.

Theorem 4. *Let $k(t)$ be a non-negative function, absolutely integrable over $0 \leqslant t \leqslant 1$, and such that $\int_0^1 k(t)\, dt > 0$. Then, if $K(x,t) \geqslant k(t)$ for all $x \geqslant 0$, $0 \leqslant t \leqslant 1$, then $K(x,t)$ is stabilizing.*

Proof. We first assume that $k(t)$ is continuous; the general case can be derived from this one. Then there are numbers a, b, C $(0 \leqslant a < b < 1,\ C > 0)$, such that

$$K(x,t) \geqslant C \qquad (a \leqslant t \leqslant b \ ,\ x \geqslant 1).$$

Let the kernels R, S be defined by

$$R(x,t) = C \qquad (a \leqslant t \leqslant b \ ,\ x \geqslant 1),$$
$$R(x,t) = 0 \qquad (t < a \text{ and } t > b \ ;\ x \geqslant 1).$$

and $S(x, t) = K(x,t) - R(x,t)$. Both R and S are non-negative. We shall write

$$\int_0^\infty R(x,t)\, f(x-t)\, dt = Rf,$$

etc. Now we have, by incomplete iteration,

$$\begin{aligned} f &= Rf + Sf = R(Rf+Sf)+Sf = \\ &= R^3 f + R^2 Sf + RSf + Sf = \\ &= R^m f + R^{m-1} Sf + R^{m-2} Sf + \dots + Sf = \\ &= R^m f + T_m f. \end{aligned}$$

Here the kernel R^m does not depend on x; it is a continuous function of t for $ma \leqslant t \leqslant mb$ and it is positive for $ma < t < mb$. The kernel $T_m(x,t)$ vanishes for $t > B$ where $B = (m-1)b+1$.

If m is large enough, $m > m_0$ say, the union of the intervals (a, b), $(2a, 2b), \dots, (ma, mb)$ is a set of measure $> \frac{7}{8} B$. Hence for $m > m_0$, the kernel

$$U = m^{-1}(R + T_1 + R^2 + T_2 + \dots + R^m + T_m)$$

has the following properties:

$$U(x,t) \geqslant 0 \quad (x \geqslant B \ , \ t \geqslant 0) \quad , \quad U(x,t) = 0 \quad (x \geqslant B \ , \ t > B)$$

$$\int_0^B U(x,t)\,dt = 1$$

$$f(x) = \int_0^B U(x,t)\, f(x-t)\,dt \qquad (x \geqslant B)$$

and finally, there is a positive constant c such that

$$\int_E U(x,t)\,dt > c$$

for any subset $E \subset (0, B)$ of measure $\mu(E) \geqslant \frac{1}{5} B$, and for all $x \geqslant B$. Now writing

$$K^*(x,t) = B^{-1}\, U(Bx, Bt) \quad , \quad f(x) = g(Bx),$$

we obtain $g = K^* g$, and theorem 3 can be applied. K^* is stabilizing, and hence K has the same property.

If $k(t)$ is not continuous, we may take it to be bounded (otherwise deal with Max $\{1, k(t)\}$ instead of $k(t)$).

If $Q^{(2)}$ is the first iteration of K, we have

$$Q^{(2)}(x,t) \geqslant \int_0^1 k(s)\, k(t-s)\,ds = k_1(t) \qquad (0 \leqslant t \leqslant 2 \ , \ x \geqslant 1)$$

The function $k_1(t)$ is continuous. We can now apply the result we just proved; this asserts that the kernel $Q^{(2)}(2x, 2t) = K^{**}(x,t)$ is stabilizing. It follows that K has the same property, and the theorem is proved.

It is not difficult to prove, under the assumptions of theorem 4, that there is a positive constant A, depending on $k(t)$ only, such that for any regular solution we have

$$f(x) - \lim_{t\to\infty} f(t) = O(e^{-Ax}) \qquad (x \geqslant 1). \tag{3.5}$$

In order to get a simple result we did not, in the above theorem, exhaust the full strength of theorem 3. Namely, we only applied it for $\eta_n =$ constant. Furthermore, we also neglected the possibility of allowing m to tend slowly to infinity, as $x \to \infty$. But it is, however, very difficult to embody the results of these possibilities in a small number of theorems.

4. *Examples.*

a. If $K(x, t)$ does not depend on x, theorem 4 can be applied immediately. Furthermore every solution $f(x)$ is continuous for $x \geqslant 1$, and, if $m \leqslant f(x) \leqslant M$ on $0 \leqslant x \leqslant 1$ almost everywhere, we have

$$m \leqslant \lim_{x \downarrow 1} f(x) \leqslant M.$$

It follows that $f(x)$ is regular (cf. the proof of theorem 1).

We state the result obtained after a simple transformation in the following form:

Theorem 5. *Let $k(t)$ be absolutely integrable and non-negative for $0 \leqslant t \leqslant 1$, and assume that λ is the real root of the equation*

$$\int_0^1 k(t)\, e^{-\lambda t}\, dt = 1. \tag{4. 1}$$

Let $f(x)$ be measurable and bounded over every finite range $0 \leqslant x \leqslant a$, and assume that

$$f(x) = \int_0^1 k(t)\, f(x-t)\, dt \qquad (x \geqslant 1).$$

Then $f(x)$ is of the form

$$f(x) = C\, e^{\lambda x} + O\,(e^{(\lambda - A) x}),$$

where A is a positive number depending on k only (cf. (3. 5)).

b. The following functional equation arises in connection with a certain prime number problem [2]). We shall devote a separate paper to it, but here we shall derive the things we can deduce from the present results. The equation is

$$x\, F(x) = \int_0^1 F(x-t)\, dt \qquad (x > 1). \tag{4. 2}$$

We can construct a solution $F_0(x)$ of (4. 2) which is continuous, positive and non-increasing for $x \geqslant 0$. Such a function can be constructed, for instance, by taking

$$F_0(x) = 1 \quad (0 \leqslant x \leqslant 1) \quad ; \quad x\, F_0'(x) = -F_0(x-1) \quad (x \geqslant 1).$$

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[2]) S. D. Chowla and T. Vijayaraghavan, J. Indian Math. Soc. (N.S.) **11**, 31–37 (1947).

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Let $F(x)$ be an arbitrary solution of (4. 2) which is bounded and measurable for $0 \leqslant x \leqslant a$ (any a). It is easily seen to be continuous for $x \geqslant 1$. Hence $f(x) = F(x)/F_0(x)$ is a regular solution of

$$f(x) = \int_0^1 \frac{F_0(x-t)}{x\,F_0(x)}\, f(x-t)\, dt. \tag{4. 3}$$

It is easily seen that the kernel

$$K(x,t) = \frac{F_0(x-t)}{x\,F_0(x)} \qquad (0 \leqslant t \leqslant 1) \tag{4. 4}$$

satisfies (1. 2). For $0 \leqslant t \leqslant 1$ we have, since F_0 is positive and decreasing, $K(x,t) \geqslant x^{-1}$. Therefore, we apply theorem 3, with $\eta_n = \gamma/(n+2)$. It follows that $f(x)$ tends to a limit as $x \to \infty$. Moreover from (2. 11) we infer

$$f(x) - \lim_{t\to\infty} f(t) = O(x^{-\frac{1}{2}}),$$

although the O-term is by far not the best possible. So any solution of (4. 2) is of the form

$$F(x) = \{C + O(x^{-\frac{1}{2}})\}\, F_0(x).$$

Mathematisch Instituut der
Technische Hogeschool te Delft.