MATHEMATICS

NEW RESULTS IN THE THEORY OF C-UNIFORM DISTRIBUTION

BY

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§ 1.   *Introduction.*

In a recent paper [1] B. MEULENBELD developed the theory of the C-uniform distribution (mod 1) of the values of a function of n variables. For the definition of this kind of distribution we refer to the paper mentioned above. In this note the author also formulated a useful test to establish the behaviour of a system of m functions of n variables with regard to the C-uniform distribution (mod 1). We repeat here the C-test in the special form ($m = 1$) in which it will be used in the present paper.

C-test.

Let n be a positive integer and let F be a sequence of n-dimensional intervals:

$$(1) \qquad Q: \quad 0 \leqslant t_\mu < T_\mu \qquad (\mu = 1, ..., n),$$

where T_n and the measure of Q tend to infinity if Q runs through F.

Let $f(t) = f(t_1, ..., t_n)$ be a function, defined for all $(t) = (t_1, ..., t_n)$ of all Q.

Then it is necessary and sufficient for the C-uniform distribution (mod 1) of the function $f(t)$ in the intervals (1), that, for every integer $h \neq 0$, $f(t)$ satisfies the relation:

$$\lim \frac{1}{T_1 T_2 ... T_n} \int_0^{T_1} \int_0^{T_2} ... \int_0^{T_n} e^{2\pi i h f(t_1, t_2, ..., t_n)} \, dt_1 \, dt_2 \, ... \, dt_n = 0,$$

if Q runs through F.

In the present paper we shall prove the following Theorems.

Theorem I.

Let F be a sequence of n-dimensional intervals

$$Q: \quad 0 \leqslant t_k < T_k \qquad (k = 1, 2, ..., n),$$

where $T_k(k = 1, 2, ..., n) \to \infty$, if Q runs through F.

Let $f(t) = f(t_1, ..., t_n)$ be a function defined for all (t) of all Q.

Let $f(t)$ have first partial derivatives with respect to each t_k, with the property:

$$(2) \qquad \left| t_k \frac{\partial f(t_1, ..., t_n)}{\partial t_k} \right| < M_k \ \text{for} \ t_k \geqq \bar{t}_k \geqq 0 \ (\bar{t}_k \ \text{fixed}),$$

uniformly in $(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n)$, where the M_k are fixed positive numbers $(k = 1, \ldots, n)$.

Then $f(t)$ is not C-uniformly distributed (mod 1) in the intervals Q of F.

For $n = 1$ we get the following

Theorem II.
Let F be a sequence of intervals

$$Q: \quad 0 \leqslant t < T, \text{ with } T \to \infty.$$

Let $f(t)$ be a differentiable function, with the property:

$$|t f'(t)| < K \text{ for } t \geqslant t_0 \geqslant 0,$$

where t_0 and K are fixed numbers.

Then $f(t)$ is not C-uniformly distributed (mod 1) in the intervals Q of F.

As an immediate consequence of Theorem II we have:

If $f(t)$ $(t \geqslant 0)$ is C-uniformly distributed (mod 1), then $t f'(t)$ cannot be bounded.

This Theorem II is a considerable improvement of a theorem we proved in [2] (Theorem II).

The additional restrictive condition we made on $t f'(t)$ in the note just mentioned can be omitted.

In order to prove Theorem I we again apply the C-test, but the argumentation is quite different from that used in our previous papers. In the present note we make use of the following

Lemma.
If $\varphi(u)$ $(u \geqslant 0)$ is a function with first and second derivative, if

$$\lim_{u \to \infty} \frac{\varphi(u)}{u} = a \text{ (constant)},$$

and

$$u \varphi''(u) \text{ is bounded for } u \geqslant u_0 \text{ (fixed)} \geqslant 0,$$

then

$$\lim_{u \to \infty} \varphi'(u) = a.$$

For the proof of this lemma we refer to [3].

Theorem III.
Let F be a sequence of n-dimensional intervals

$$Q: \quad 0 \leqslant t_k < T_k \qquad (k = 1, \ldots, n),$$

where T_n and the measure of Q tend to infinity if Q runs through F.

Let $f(t) = f(t_1, \ldots, t_n)$ be a measurable function defined for all (t) of all Q.

Let $f(t)$ have a partial derivative with respect to t_n with the property:

(3) $$\lim_{t_n \to \infty} t_n^p \frac{\partial f(t_1, \ldots, t_n)}{\partial t_n} = c \neq 0,$$

uniformly in $(t_1, \ldots, t_{n-1})$, where c and p are fixed numbers, and $0 \leqslant p < 1$. Then $f(t)$ is C-uniformly distributed (mod 1) in the intervals Q of F.

For $n = 1$ we get

Theorem IV.

Let F be a sequence of intervals

$$Q: \quad 0 \leqslant t < T, \quad \text{with} \quad T \to \infty.$$

Let $f(t)$ $(t \geqslant 0)$ be a differentiable function with the property

$$\lim_{t \to \infty} t^p f'(t) = c \neq 0, \quad 0 \leqslant p < 1.$$

Then $f(t)$ is C-uniformly distributed (mod 1) in the intervals Q of F.

This Theorem is a generalisation of Theorem III of [2], where we assumed $p = 0$. N. H. KUIPER reported us by letter that he also possesses a proof of Theorem IV.

In § 2 we prove Theorem I, in § 3 Theorem III, while in § 4 we give some examples.

We remark that Theorem IV does not hold if we take $p = 1$.

In this case $f(t)$ is not C-uniformly distributed (mod 1) as follows from Theorem II.

§ 2. *Proof of Theorem I.*

We shall show that the expression

$$I = \frac{1}{T_1 \ldots T_n} \int_0^{T_1} \ldots \int_0^{T_n} e^{2\pi i h f(t_1, \ldots, t_n)} \, dt_1 \ldots dt_n$$

does not tend to zero, if Q runs through F.

Let us suppose that

$$I^* = \frac{1}{T_1 \ldots T_n} \int_0^{T_1} \ldots \int_0^{T_n} \cos 2\pi h f(t_1, \ldots, t_n) \, dt_1 \ldots dt_n$$

tends to zero if Q runs through F. Then we should also have:

$$\lim_{T_n \to \infty} \ldots \lim_{T_1 \to \infty} \frac{1}{T_1 \ldots T_n} \int_0^{T_1} \ldots \int_0^{T_n} \cos 2\pi h f(t_1, \ldots, t_n) \, dt_1 \ldots dt_n =$$

$$\lim_{T_n \to \infty} \ldots \lim_{T_2 \to \infty} \left[\lim_{T_1 \to \infty} \frac{1}{T_1} \int_0^{T_1} \left\{ \frac{1}{T_2 \ldots T_n} \int_0^{T_2} \ldots \int_0^{T_n} \cos 2\pi h f(t_1, \ldots, t_n) \, dt_2 \ldots dt_n \right\} dt_1 \right] =$$

$$\lim_{T_n \to \infty} \ldots \lim_{T_2 \to \infty} \left[\lim_{T_1 \to \infty} \frac{1}{T_2 \ldots T_n} \int_0^{T_2} \ldots \int_0^{T_n} \cos 2\pi h f(T_1, t_2, \ldots, t_n) \, dt_2 \ldots dt_n \right] =$$

(as follows from the lemma and from (2) with $k = 1$)

$$\lim_{T_1\to\infty} {}'\lim_{T_n\to\infty} \ldots \lim_{T_2\to\infty} \frac{1}{T_2} \int_0^{T_2} \left\{ \frac{1}{T_3 \ldots T_n} \int_0^{T_3}\ldots\int_0^{T_n} \cos 2\pi h f(T_1, t_2,\ldots,t_n)\, dt_3 \ldots dt_n \right\} dt_2 =$$

$$\lim_{T_1\to\infty} \lim_{T_n\to\infty} \ldots \lim_{T_2\to\infty} \frac{1}{T_3 \ldots T_n} \int_0^{T_3}\ldots\int_0^{T_n} \cos 2\pi h f(T_1, T_2, t_3,\ldots,t_n)\, dt_3 \ldots dt_n = 0$$

(as follows again from the lemma and from (2) with $k = 2$).

Repeating this argument we should finally have:

$$(4) \qquad\qquad \lim_{T_1\to\infty} \ldots \lim_{T_n\to\infty} \cos 2\pi h f(T_1,\ldots,T_n) = 0.$$

If we furthermore suppose that

$$I^{**} = \frac{1}{T_1 \ldots T_n} \int_0^{T_1} \ldots \int_0^{T_n} \sin 2\pi h f(t_1,\ldots,t_n)\, dt_1 \ldots dt_n$$

tends to zero if Q runs through F, then we should find in a similar way:

$$(5) \qquad\qquad \lim_{T_1\to\infty} \ldots \lim_{T_n\to\infty} \sin 2\pi h f(T_1,\ldots,T_n) = 0.$$

Both relations (4) and (5) however cannot be satisfied simultaneously. So our assumption is false, and we see that I does not tend to zero if Q runs through F.

§ 3. *Proof of Theorem III.*

Put $P = T_1 T_2 \ldots T_n$. Without loss of generality we assume $c > 0$. From (3) it follows, that, for an arbitrary small $\varepsilon > 0$ and $t_n > T_n^* = T_n^*(\varepsilon)$, we have

$$(6) \qquad\qquad \left| \frac{1}{t_n^p \dfrac{\partial f(t_1,\ldots,t_n)}{\partial t_n}} - \frac{1}{c} \right| < \varepsilon,$$

uniformly in $(t_1, \ldots, t_{n-1})$.

For $T_n > T_n^*$ the expression

$$I = \frac{1}{P} \int_0^{T_1} \ldots \int_0^{T_n} e^{2\pi i h f(t_1,\ldots,t_n)}\, dt_1 \ldots dt_n$$

can be written as

$$(7) \qquad\qquad I = \frac{1}{P} \int_0^{T_1} \ldots \int_0^{T_n^*} + \frac{1}{P} \int_0^{T_1} \ldots \int_{T_n^*}^{T_n}.$$

It is easily seen that the first term on the right of (7) tends to zero if Q

runs through F. The second term on the right of (7) equals

$$(8) \quad \begin{cases} \dfrac{1}{P} \displaystyle\int_0^{T_1} \cdots \int_0^{T_{n-1}} \left[\int_{f(T_n^*)}^{f(T_n)} \dfrac{e^{2\pi i h u}\, du}{\dfrac{\partial f(t_1,\ldots,t_n)}{\partial t_n}} \right] dt_1 \ldots dt_n = \\[3em] \dfrac{1}{P} \displaystyle\int_0^{T_1} \cdots \int_0^{T_{n-1}} \left[\int_{f(T_n^*)}^{f(T_n)} t_n^p\, e^{2\pi i h u} \left(\dfrac{1}{t_n^p \dfrac{\partial f(t_1,\ldots,t_n)}{\partial t_n}} - \dfrac{1}{c} \right) du \right] dt_1 \ldots dt_{n-1} + \\[3em] \dfrac{1}{cP} \displaystyle\int_0^{T_1} \cdots \int_0^{T_{n-1}} \left[\int_{f(T_n^*)}^{f(T_n)} \{F(u)\}^p\, e^{2\pi i h u}\, du \right] dt_1 \ldots dt_{n-1}, \end{cases}$$

where we put, for the sake of brevity,

$$f(T_n) = f(t_1,\ldots,t_{n-1},T_n)$$
$$f(T_n^*) = f(t_1,\ldots,t_{n-1},T_n^*),$$

and where $t_n = F(u,t_1,\ldots,t_{n-1})$ is the inverse function of $u = f(t_1,\ldots,t_n)$. This inverse function exists on account of our assumption $c > 0$.

The first term on the right of (8) is in absolute value less than

$$(9) \qquad \varepsilon \left| \frac{f(T_n) - f(T_n^*)}{T_n} \right| T_n^p < \varepsilon \frac{|f(T_n)| + |f(T_n^*)|}{T_n^{1-p}}.$$

From

$$\lim_{T_n \to \infty} \frac{|f(T_n)|}{T_n^{1-p}} = \lim_{T_n \to \infty} \frac{T_n^p |f'(T_n)|}{1-p} = \frac{c}{1-p},$$

(9), and the assumption $0 \leqslant p < 1$, it follows, that the first term on the right of (8) tends to zero if Q runs through F.

Furthermore we have:

$$\int_{f(T_n^*)}^{f(T_n)} \{F(u)\}^p\, e^{2\pi i h u}\, du = \left[\frac{\{F(u)\}^p\, e^{2\pi i h u}}{2\pi i h} \right]_{u=f(T_n^*)}^{u=f(T_n)} +$$

$$- \frac{p}{2\pi i h} \int_{f(T_n^*)}^{f(T_n)} e^{2\pi i h u} \{F(u)\}^{p-1}\, F'(u)\, du = \frac{T_n^p\, e^{2\pi i h f(T_n)}}{2\pi i h} - \frac{T_n^{*p}\, e^{2\pi i h f(T_n^*)}}{2\pi i h} +$$

$$- \frac{p}{2\pi i h} \int_{T_n^*}^{T_n} e^{2\pi i h f(t)}\, t_n^{p-1}\, dt_n = K_1 + K_2 + K_3,\ \text{say}.$$

Now we have, replacing the form [ ] in the second term on the right

of (8) by $K_1 + K_2 + K_3$, successively

$$\left| \frac{1}{cP} \int\limits_{0}^{T_1} \dots \int\limits_{0}^{T_{n-1}} K_1 \, dt_1 \dots dt_{n-1} \right| < \frac{1}{2\pi \, |h| \, c \, T_n^{1-p}},$$

$$\left| \frac{1}{cP} \int\limits_{0}^{T_1} \dots \int\limits_{0}^{T_{n-1}} K_2 \, dt_1 \dots dt_{n-1} \right| < \frac{T_n^{*p}}{2\pi \, |h| \, c \, T_n},$$

$$\left| \frac{1}{cP} \int\limits_{0}^{T_1} \dots \int\limits_{0}^{T_{n-1}} K_3 \, dt_1 \dots dt_{n-1} \right| < \frac{T_n^p - T_n^{*p}}{2\pi \, |h| \, c \, T_n};$$

and from these inequalities it follows that also the second term on the right of (8) tends to zero if Q runs through F.

This completes the proof.

§ 4. *Examples.*

a) The functions

$$f(t) = \lg(1 + t_1 + \dots + t_n),$$

$$f(t) = \sum_{k=1}^{n} \lg(1 + t_k),$$

$$f(t) = \lg\left(a_0 + \sum_{k=1}^{n} a_k t_k^{\beta_k}\right) \text{ with } a_0 > 0 \text{ and } a_k, \beta_k > 0 \quad (k = 1, \dots, n),$$

are not C-uniformly distributed (mod 1) in the intervals

(10) $\qquad\qquad 0 \leqslant t_k < T_k, T_k \to \infty \qquad (k = 1, \dots, n),$

as follows from Theorem I.

b) The function

$$f(t) = \sum_{k=1}^{n} t_k^{1-p} + \sum_{k=1}^{n} \frac{\sin t_k}{t_k} + \psi(t_1, \dots, t_{n-1}),$$

where ψ is an arbitrary real measurable function, and where $0 \leqslant p < 1$, satisfies

$$\lim_{t_n \to \infty} t_n^p \frac{\partial f(t_1, \dots, t_n)}{\partial t_n} = 1 - p > 0, \text{ uniformly in } (t_1, \dots, t_{n-1}),$$

so that $f(t)$ is C-uniformly distributed (mod 1) in the intervals (10), as follows from Theorem III.

Bandung, 1 April 1950. *University of Indonesia.*

LITERATURE

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