

MATHEMATICS

ON PARTIALLY ORDERED GROUPS

BY

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1. The present paper contains a discussion of order-homomorphisms, including certain further strongly related concepts and theorems on partially ordered groups, some of which are of interest in themselves too.

After collecting certain known definitions and facts needed throughout the paper, we define order-homomorphisms as group-homomorphisms preserving order. The kernels and inducers of order-homomorphisms are proved to be the convex invariant subgroups. A new and fundamental notion is that of tautomorphism which is a group-isomorphism preserving order in *one* direction. The necessity of introducing this concept may be seen in view of Theorem 2. But there is a basic difficulty connected with the notion of tautomorphism, since it is not symmetric and therefore the theorems corresponding to the isomorphism-theorems are not enough deep for proving the JORDAN-HÖLDER theorem for partially ordered groups.

For the proofs we remark that the theorems on order-homomorphisms include two essentially different propositions: one concerning group-homomorphism and one concerning order-preserving. Since the pure group-theoretic parts of the theorems are familiar facts ¹⁾, we may and shall omit them and shall consistently confine ourselves to discussing the order-preserving.

A new concept analogous to the KANTOROVITCH-RIESZ-BIRKHOFF l -ideals ²⁾ is introduced, it is defined in terms of a generalized "absolute" introduced in my paper [5]. It is shown that these ideals are convex invariant subgroups with the MOORE-SMITH property. We finally give an interesting theorem on ideals in abelian, normally ordered groups.

2. We begin by recalling a few definitions and facts on which the sequel depends.

A group G is said to be *partially ordered* (p.o.) if an order relation \geq is defined for some pairs of elements in G satisfying ³⁾ (i) *reflexiveness*:

¹⁾ Cf. e.g. ZASSENHAUS [9]. Numbers in brackets refer to the bibliography given at the end of the paper.

²⁾ KANTOROVITCH [7], RIESZ [8], BIRKHOFF [1].

³⁾ See e.g. EVERETT and ULAM [4].

$x \geq x$ for every x in G , (ii) *antisymmetry*: $x \geq y$ and $y \geq x$ imply $x = y$, (iii) *transitivity*: $x \geq y$ and $y \geq z$ imply $x \geq z$, (iv) *the MOORE-SMITH property*: for every pair x, y there exists a z with $z \geq x$, $z \geq y$, finally, (v) *homogeneity*: $x \geq y$ implies $u + x + v \geq u + y + v$ for every u, v in G . (The group-operation is written as addition, but we do not require commutativity.) We shall say x and y *incomparable*, $x \parallel y$, if neither $x \geq y$, nor $y \geq x$.

The elements t satisfying $t \geq 0$ (where 0 is the group identity) are called *positive*. Homogeneity implies that $x \geq y$ if and only if $x - y$ (or $-y + x$) is positive; therefore the definition of a partially ordered group may also be given in terms of "positiveness" in G , for example, the *MS-property* is equivalent to requiring that each element be the difference of two suitable positive elements (CLIFFORD [3]).

We call G *simply* or *linearly ordered* if $x \parallel y$ is impossible. If a p.o. group G is at the same time a lattice under the same order relation \geq , we say G is a *lattice-ordered group*. By an *archimedean-ordered group* we mean a p.o. group such that $nx < y$, $n = 0, \pm 1, \pm 2, \dots$ implies $x = 0$.

The partial order defined in G is said to be *normal* (FUCHS [5], [6]) if $nx \geq 0$ for some positive integer n implies $x \geq 0$. It is immediate that linear order is normal and normality implies that in the group every element except 0 has an infinite order.

If two partial orders P and R are defined on the same group and if $x \geq y$ in P always implies $x \geq y$ in R , it is natural to say that R is an *extension* of P . It is convenient to consider the same group G with different partial orders P and R as distinct p.o. groups, $G(P)$ and $G(R)$. If R is an extension of P , we shall say $G(R)$ an *order-extension* (briefly *o-extension*) of $G(P)$.

3. Let G and H be two p.o. groups. A single-valued mapping η of G onto H is called an *order-homomorphism* (*o-homomorphism*) if it is a group-homomorphism which preserves order, i.e., $\eta(x + y) = \eta(x) + \eta(y)$, and $x \geq y$ implies $\eta(x) \geq \eta(y)$. A group-isomorphism Θ meeting the requirement " $x \geq y$ implies $\Theta(x) \geq \Theta(y)$ and vice-versa" will be called an *o-isomorphism*. The definitions of *o-automorphisms* as well as of *o-endomorphisms* are now obvious.

The existence of order-preserving transformations is a consequence of the simple fact that the inner automorphisms preserve order in both directions. In fact, from homogeneity we conclude that $x \geq y$ if and only if $-a + x + a \geq -a + y + a$.

It may happen that G and H are isomorphic groups in the pure group-theoretical sense, and G is *o-homomorphic* to H , but they fail to be *o-isomorphic*. In this case we say G *tautomorphic* to H . It is evident that each group is tautomorphic to any of its *o-extensions*. Moreover, it may readily be verified that G is tautomorphic to H if, and only if, G has an *o-extension* *o-isomorphic* to H .

Since a linear order has no proper o -extension, it follows that if G is tautomorphic to H and G is linearly ordered, then G is necessarily o -isomorphic to H .

By a *convex subgroup* of G is meant a subgroup C containing with any x, y ($x \geq y$) all elements between x and y . If α is any o -automorphism of G , then the convexity of C implies that of $\alpha(C)$, and conversely. Hence all conjugate groups of a convex subgroup are convex.

4. Recalling the definition of the *kernel* of a mapping as the set of elements sent into 0, we state the fundamental theorem⁴⁾ on o -homomorphisms:

Theorem 1. *The kernel of an o -homomorphism is a convex invariant subgroup, and every convex invariant subgroup C of G induces an o -homomorphic mapping of G upon the factor-group G/C .*

The convexity of the kernel follows from the definition of o -homomorphisms, according to which together with x, y all elements between x and y are mapped upon 0. (Hence it is clear that an o -homomorphism is a tautomorphism if, and only if, the kernel consists of 0 alone).

For the proof of the converse we define a natural partial order in the factor group G/C by the specification that for two cosets we put $C+x \geq C+y$ if and only if for some representatives $u \in C+x$, $v \in C+y$ a relation $u \geq v$ holds. To justify this definition of order, one has to verify the fulfilment of all conditions for partial orders listed in 2. These conditions are partly automatically satisfied, partly may be checked readily, so that there is no need of a detailed demonstration⁵⁾. However, for the proof as well as for later discussions it is useful to remark that the definition of the natural order in G/C may be expressed in the apparently more stricter but clearly equivalent form too: $C+x \geq C+y$ if and only if to every u in $C+x$ there is a v in $C+y$ such that $u \geq v$. Whenever we are speaking of G/C as a p.o. group we always mean G/C with the natural order induced by the order of G .

Now the definition of the partial order in G/C at once establishes that the mapping $x \rightarrow C+x$ of G onto G/C is an o -homomorphism, indeed.

The natural order in G/C is the worst possible one in the sense of

Theorem 2. *H is an o -homomorphic image of G if, and only if, the factor-group G/C is tautomorphic to H , where C is the kernel of the mapping $G \rightarrow H$.*

Since by Theorem 1 G is o -homomorphic to G/C , G is necessarily

⁴⁾ The first part of this theorem is a special case of a general theorem on order-preserving mappings of partially ordered sets.

⁵⁾ For example, (ii) may be verified as follows. $C+x \geq C+y$ and $C+y \geq C+x$ imply $x^* \geq y \geq x^{**}$ for some x^*, x^{**} in $C+x$. Hence by homogeneity we conclude $x^* - x^{**} \geq y - x^{**} \geq 0$ with $x^* - x^{**} \in C$, so that by convexity we are directly led to $y - x^{**} \in C$, i.e., $C+y = C+x^{**} = C+x$.

ϕ -homomorphic to H , whenever G/C is tautomorphic to H . Conversely, if G is ϕ -homomorphic to H under the mapping $x \rightarrow x^*$ ($x \in G$, $x^* \in H$) with the kernel C , then consider the mapping $C + x \rightarrow x^*$ of G/C upon H . This is a group-isomorphism preserving order. $C + x \geq C + y$ implies $u \geq v$ for some $u \in C + x$, $v \in C + y$, hence we get $u^* \geq v^*$, which proves the statement, because $u^* = x^*$ and $v^* = y^*$.

5. In this section we are interested in studying the tautomorphism theorems corresponding to the well-known isomorphism theorems in group theory. As was already emphasized in the introduction, we must discuss the propositions only from the point of view of order-preserving.

Theorem 3. (*First tautomorphism theorem.*) Assume that $G \rightarrow H$ is an ϕ -homomorphism and D is an invariant convex subgroup of H . Then the subgroup C of G consisting of all elements mapped upon D is an invariant convex subgroup; moreover, G/C is tautomorphic to H/D .

From Theorem 1 it follows that G is ϕ -homomorphic to H/D ; the kernel of this transformation is evidently the totality of elements sent into D , that is C . This fact establishes the invariance and convexity of C and by Theorem 2 one is immediately led to the tautomorphism of G/C to H/D .

Theorem 4. (*Second tautomorphism theorem.*) If U, C are subgroups of the p.o. group G , and C is invariant convex in G , then $U \cap C$ is an invariant convex subgroup of U and $U/U \cap C$ is tautomorphic to $U + C/C$.

U is obviously ϕ -homomorphic to the factor group $U + C/C$ under the correspondence $x \rightarrow C + x$, x in U . Since the elements mapped onto 0 are those of $U \cap C$, Theorems 1 and 2 prove all assertions of our theorem.

At this stage it is natural to try to prove the analogue of SCHREIER's theorem expressing the fact that any two normal series of a group have equivalent refinements. But in the present case there are insurmountable difficulties in the proof, caused by the asymmetric character of tautomorphism and by the fact that $C + D$ need not be invariant convex if so are C and D ⁶⁾. So that our result would be extremely weak, practically expressing nothing at all, therefore we shall omit it.

6. It is an elementary fact that any finite group without proper subgroups is a cyclic group of prime order, hence is isomorphic to the additive group of integers modulo a prime and is commutative. We may

⁶⁾ This may be illustrated by the example of the additive group of all real two-dimensional vectors; we put $(a, b) \geq (c, d)$ if and only if \geq componentwise. Let a, b, c, d be positive numbers such that $a/c > b/d$ and choose a rational number p/q with $a/c > p/q > b/d$. The elements of the type $(-ka, kb)$ as well as those of the type $(-kc, kd)$ with integer k constitute discrete convex subgroups whose union-group is clearly not convex, since $(-qa, qb) < (-pc, pd)$ contradicts discreteness.

state something similar on certain p.o. groups S without proper convex subgroups:

Theorem 5. *Let S be a group with a normal partial order having no proper convex subgroups. Then S is isomorphic to a subgroup of the additive group of real numbers ordered by magnitude, hence is commutative.*

At first, S is *linearly ordered*. For, assuming the contrary we can find an element $x \parallel 0$ and normality implies $nx \parallel 0$ for $n = \pm 1, \pm 2, \dots$. It is readily seen that the cyclic subgroup generated by x is convex, hence is identical to S . This case is impossible, since S now fails to possess the *MS*-property.

Further, S is *archimedean*. Indeed, $nx < y$, $n = 0, \pm 1, \pm 2, \dots$ implies that y does not belong to the least convex subgroup containing x , consequently, $x = 0$.

By a theorem due to H. CARTAN⁷⁾ our theorem is completely proved.

7. In lattice-ordered groups the homomorphisms (with respect both to group- and lattice-operations) can be described by the so-called *l-ideals* defined as invariant subgroups containing with any a also all x such that $|x| \leq |a|$ ⁸⁾. Evidently, this definition has no meaning in p.o. groups that are not lattice-ordered. But if we appropriately define the concept of "absolute" $\|x\|$, viz. as the set of all upper bounds for x and $-x$ ⁹⁾, we may define the corresponding concept: an invariant subgroup I of G will said to be an *ideal*, if 1) it contains with any a also all x with¹⁰⁾ $\|x\| \supset \|a\|$, and 2) it satisfies the *MS*-property¹¹⁾.

Ideals are expected to be closely connected with convex subgroups. Actually:

Theorem 6. *A set I is an ideal if, and only if, it is an invariant convex subgroup with the *MS*-property.*

Before entering into the proof we observe that for any subgroup with the *MS*-property convexity is equivalent to the property of containing with a also all elements between a and $-a$. For, to any x, y (say $x \geq y$) in the subgroup C we can find by the *MS*-property an a in C such that $a \geq x$ and $a \geq -y$, and if C contains each element between a and $-a$, then it a fortiori contains each element between x and y , being $a \geq x \geq y \geq -a$.

Now let x lie between a and $-a$, $a \geq x \geq -a$. Then $\|x\|$ contains a , by definition, and hence contains the set of elements $\geq a$, which is now

⁷⁾ CARTAN [2]: "A linearly ordered archimedean group is isomorphic to a subgroup of the additive group of all real numbers".

⁸⁾ By the *absolute* $|x|$ is meant the join of x and $-x$, $x \cup -x$. Cf. e.g. BIRKHOFF [1].

⁹⁾ For its definition and its main properties see my paper [5].

¹⁰⁾ \supset is the sign of inclusion.

¹¹⁾ Given $x, y \in I$, some $z \in I$ satisfies $z \geq x$ and $z \geq y$.

plainly equal to $\|a\|$; that is, $\|x\| \supset \|a\|$ and therefore all ideals are convex.

Conversely, let a belong to a convex invariant subgroup C with the *MS*-property. Assume $\|x\| \supset \|a\|$ and take a c in C such that $c \geq a$, $c \geq -a$. Then $\|a\|$ and hence $\|x\|$ contains c , consequently, x lies between c and $-c$. By convexity we conclude that C is actually an ideal.

8. Let G be a commutative group with a normal partial order P and I an ideal of $G(P)$. If we extend P to a linear order L of G and at the same time adjoin to I all x satisfying a relation $a \geq x \geq -a$ in L ($a \in I$), then I becomes an ideal $I(L)$ of $G(L)$. We shall call $I(L)$ briefly a *linear extension of $I(P)$* .

Theorem 7. *Any ideal I in an abelian group with a normal partial order P is the intersection of its linear extensions $I(L)$.*

The proof of this theorem is based on the following lemmas.

Lemma 1. *If P is a normal partial order on a commutative group G and x, y are any two elements incomparable in P , then there exists a normal extension R_{xy} of P such that $x \geq y$ in R_{xy} .*

Indeed, putting $a \geq b$ in R_{xy} if and only if $p(a - b) \geq q(x - y)$ in P for some non-negative integers p, q , not both zero, we get a normal partial order satisfying the conditions ¹²⁾.

Lemma 2. *If x does not belong to the ideal $I(P)$, then P has an extension R_x such that either $a \geq x$ in R_x for all $a \in I$, or dually, $x \geq a$ in R_x for all $a \in I$.*

x not in I and the convexity of I imply the impossibility of one of the relations: $x \geq a$ in P for some $a \in I$, and $b \geq x$ in P for some $b \in I$. Assuming the second relation impossible, either $x \geq a$ in P is true for all $a \in I$ (when the proof is finished), or there is a $c \in I$ with $x \parallel c$. In the latter case, using Lemma 1, let us extend P to P_c , such that $x \geq c$ in P_c . In the new order P_c a relation like $b \geq x$ for some $b \in I$ is again impossible, since by the definition of P_c this would mean $p(b - x) \geq q(x - c)$ in P , that is, $pb + qc \geq (p + q)x$ in P for some non-negative integers p, q , not both zero. Choosing a $d \in I$ such that $d \geq b$, $d \geq c$ in P , we conclude that $(p + q)d \geq (p + q)x$ in P , whence by normality $d \geq x$ in P , against the assumptions on x . By a transfinite continuation of this process of extending the order, ZORN's maximal principle [10] establishes Lemma 2.

Resuming the proof of Theorem 7, it is clear that we need only to prove that the intersection of the $I(L)$ contains no element not in I . Given x not in I , using the maximal principle again, Lemmas 1 and 2 guarantee the existence of a linear extension L_0 of P such that either $a \geq x$ in L_0 for all $a \in I$ or dually. In either case, a is not in $I(L_0)$, q.e.d.

¹²⁾ For the proof we refer to FUCHS [6].

BIBLIOGRAPHY

1. BIRKHOFF, G., Lattice-ordered groups, *Annals Math.* **43**, 298—331 (1942).
2. CARTAN, H., Un théorème sur les groupes ordonnés, *Bulletin Sci. Math.* **63**, 201—205 (1939).
3. CLIFFORD, A. H., Partially ordered abelian groups, *Annals Math.* **41**, 465—473 (1940).
4. EVERETT, C. J. and S. ULAM, On ordered groups, *Transactions Amer. Math. Soc.* **57**, 208—216 (1945).
5. FUCHS, L., Absolutes in partially ordered groups, *Proceedings Kon. Ned. Akad. v. Wetensch., Amsterdam* **52**, 251—255 (1949).
6. ———, On the extension of the partial order of groups, *Amer. Journal Math.* **72**, 191—194 (1950).
7. KANTOROVITCH, L. V., Lineare halbgeordnete Räume, *Mat. Sbornik* **44**, 121—168 (1937).
8. RIESZ, F., Sur quelques notions fondamentales dans la théorie générale des opérations linéaires, *Annals Math.* **41**, 174—206 (1940).
9. ZASSENHAUS, H., *Lehrbuch der Gruppentheorie*, vol. 1, Leipzig-Berlin (1937).
10. ZORN, M., A remark on method in transfinite algebra, *Bulletin Amer. Math. Soc.* **41**, 667—670 (1935).