MATHEMATICS

PROJECTIVE GEOMETRIZATION OF A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS, III: PROJECTIVE NORMAL SPACES

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Synopsis. This paper is a continuation of two previous papers with the same title which will be referred to as *PI* and *PII* (Kon. Ned. Akad. v. Wet., Proceedings Vol. **53,** Nos 3 and 4, 1950). In the last section of *PII* we saw that the points $\mathbf{n}$ are not appropriate to define "projective normal spaces". In this paper we use the results of *PI* and *PII* to find a set of ("privileged") points which (together with $\mathbf{x}$) may be used for the definition of projective normal spaces. These spaces are defined in the last section of this paper.

§ 1. *Introductory notions.*

Lemma (1, 1). *The normal points* (*cf.* § 4 *in PI*) *satisfy the condition*

$$\mathbf{n}_{a_r \ldots a_1} = \mathbf{n}_{(a_r \ldots a_1)}, \qquad r = 2, \ldots, N. \tag{1, 1}$$

Proof. The symbols $\left\{ P_{a'_r \ldots a'_s}^{\quad a_s} P_{a'_{s-1} \ldots a'_1}^{a_{s-1} \ldots a_1} \right\}$ and $P_{a'_r \ldots a'_s \ldots a'_1}^{\quad a_s \ldots a_1}$ which appear in (1, 7) *PI* are obviously symmetric in their subscripts. If we replace in these symbols the derivatives $P_{a'_q \ldots a'_u}^{\quad a_u}$ by $\Gamma_{a_u \ldots a_q}^{\quad b_q}$ (which are symmetric in their subscripts) and the $P^b_{a'}$ by δ^b_a we obtain the symbols $\left\{ \Gamma_{a_r \ldots a_s}^{\quad b_s} \delta_{a_{s-1} \ldots a_1}^{b_{s-1} \ldots b_1} \right\}$ and $\gamma_{a_r \ldots a_s \ldots a_1}^{\quad b_s \ldots b_1}$, symmetric in their subscripts. Consequently the normal points defined by (4, 2) *PI* satisfy (1, 1).

Another lemma to be used later on deals with the tensor $Q^b_a = y^c \Gamma^b_{ac}$ (cf. Theorem (3, 4) in *PI*).

Lemma (1, 2). *If* $N > 2$ *then*

$$y^c K_{(cba)}^{\quad d} = y^c K_{cba}^{\quad d} - \tfrac{2}{3} D_{(b} Q^d_{a)} \tag{1, 2a}$$

and consequently if

$$Q^d_a = Q\, \delta^d_a, \qquad Q = \text{const.} \tag{1, 3}$$

then

$$y^c K_{(cba)}^{\quad d} = y^c K_{cba}^{\quad d} \tag{1, 4a}$$

$$y^c D_{(c} \mathbf{n}_{ba)} = y^c D_c \mathbf{n}_{ba} \tag{1, 4b}$$

Proof. We have

$$(1, 5) \qquad -y^c \partial_c \Gamma^d_{ab} = \Gamma^d_{ab}, \quad -y^c \partial_a \Gamma^d_{bc} = \Gamma^a_{ba} - \partial_a Q^d_b.$$

On the other hand if $N > 2$, then $K_{cba}{}^d$ may be thought of as defined by (2, 3) *PII* and this equation together with (1, 5) leads at once to (1, 2a), from which (1, 4a) follows by virtue of (1, 3). Using the equation (2, 2b) in *PII*, we see that

$$(1, 6) \qquad y^c D_c \mathbf{n}_{ba} = y^c \mathbf{n}_{cba} + y^c K_{cba}^{\ \ d} \mathbf{x}_d$$

$$(1, 7) \qquad y^c D_{(c} \mathbf{n}_{ba)} = y^c \mathbf{n}_{(cba)} + y^c K_{(cba)}^{\ \ \ d} \mathbf{x}_d.$$

The equation (1, 4b) follows from (1, 6), (1, 7), (1, 1) and (1, 4a).

In the following definition we use the projective tensors K's (cf. § 1 in *PII*):

Definition (1, 1). *A $\mathfrak{P}_m$ will be referred to as symmetric if the following conditions hold:*

I) *The tensor Q^b_a satisfies the relation*

$$(1, 8a) \qquad Q^b_a = Q\, \delta^b_a$$

where $Q \neq -1$ is a constant.

II) *Among the tensors K's there is at least one, say $K_{a_{u+1}\ldots a_1}{}^b$, such that*

$$(1, 8b) \qquad K \equiv K_{a_{u+1}\ldots a_1}{}^{a_{u+1}}\, y^{a_u} \ldots y^{a_1} \neq 0.$$

III) *If $N > 3$ then* [1])

$$(1, 9) \quad \begin{cases} a) \quad y^{a_r} D_{a_r} \mathbf{n}_{a_{r-1}\ldots a_1} = y^{a_r} D_{(a_r} \mathbf{n}_{a_{r-1}\ldots a_1)} \ ^{2)} \qquad (r = 4, \ldots, N,\ \mathbf{x}_a = \mathbf{n}_a) \\ b) \quad y^{a_p} D_{a_p} D_{(a_{p-1}} \ldots D_{a_{s+1}} K_{a_s \ldots a_1)}{}^c = \\ \qquad y^{a_p} D_{(a_p} \ldots D_{a_{s+1}} K_{a_s \ldots a_1)}{}^c \qquad (s = 3, 4, \ldots, N-1,\ p = s+1, \ldots, N). \end{cases}$$

Throughout this paper we will deal with symmetric cases only without stating it explicitly.

One of the consequences of (1, 8a) and (1, 9a) is stated in the following lemma where we put

$$(1, 10) \qquad c_r \equiv N + 1 - (r-1)(Q+1), \qquad r = 2, \ldots, N.$$

Lemma (1, 3). *If $N \geqq 2$, then*

$$(1, 11a) \qquad y^b \mathbf{n}_{ba} = y^b \mathbf{n}_{(ba)} = c_2\, \mathbf{x}_a.$$

[1]) The condition (1, 9a) is satisfied for $r = 2$ and if $N > 2$ also for $r = 3$ (cf. the equation (1, 4b)).

[2]) We impose this condition in order to simplify the final results. The device used later on (cf. the equations (1, 12)) may easily be generalized for the case where (1, 9a) does not hold.

If $N \geqq 3$ *then*

(1, 11*b*) $$y^c \mathbf{n}_{cba} = y^c \mathbf{n}_{(cba)} = c_3 \mathbf{n}_{ba} - y^c K_{(cba)}^{\ \ d} \mathbf{x}_d.$$

Moreover if $N \geqq 4$ *then*

(1, 11*c*) $$\begin{cases} y^{a_r} \mathbf{n}_{a_r \dots a_1} = y^{a_r} \mathbf{n}_{(a_r \dots a_1)} = c_r \mathbf{n}_{a_{r-1} \dots a_1} - \\ - \sum_2^{r-2} {}_s\, y^{a_r} \left\{ \left\{ K_{(a_r \dots a_s}^{\ \ b_s} \delta_{a_{s-1} \dots a_1)}^{b_{s-1} \ \ b_1} \right\} \right\} \mathbf{n}_{b_s \dots b_1} - y^{a_r} K_{(a_r \dots a_1)}^{\ \ b} \mathbf{x}_b, \quad r = 4, \dots, N. \end{cases}$$

Proof: The first equations (1, 11) follow at once from (1, 1) and

$$\mathbf{n}_{ba} = \mathbf{x}_{ba} - \Gamma_{ba}^{c} \mathbf{x}_c = \mathbf{n}_{ba}$$

Moreover we have by virtue of (1, 8*a*)

(1, 12*a*) $$\begin{cases} y^{a_r} D_{a_r} \mathbf{n}_{a_{r-1} \dots a_1} = y^{a_r} [\partial_{a_r} \mathbf{n}_{a_{r-1} \dots a_1} - \Gamma_{a_{r-1} a_r}^{h} \mathbf{n}_{h a_{r-2} \dots a_1} - \dots - \Gamma_{a_1 a_r}^{h} \mathbf{n}_{a_{r-1} \dots a_2 h}] \\ \qquad = [N + 1 - (r-1)] \mathbf{n}_{a_{r-1} \dots a_1} - Q(r-1) \mathbf{n}_{a_{r-1} \dots a_1} = c_r \mathbf{n}_{a_{r-1} \dots a_1}. \\ \qquad r = 3, \dots, N \end{cases}$$

On the other hand if we use (2, 2) in *PII*, (1, 9*a*), (1, 1) and (1, 4) we obtain

(1, 12*b*) $$\begin{aligned} & y^{a_r} [\mathbf{n}_{a_r \dots a_1} - D_{a_r} \mathbf{n}_{a_{r-1} \dots a_1}] = y^{a_r} [\mathbf{n}_{(a_r \dots a_1)} - D_{(a_r} \mathbf{n}_{a_{r-1} \dots a_1]} = \\ & - \sum_2^{r-2} {}_s\, y^{a_r} \left\{ \left\{ K_{(a_r \dots a_s}^{\ \ b_s} \delta_{a_{s-1} \dots a_1)}^{b_{s-1} \ \ b_1} \right\} \right\} \mathbf{n}_{b_s \dots b_1}{}^{3)} - y^{a_r} K_{(a_r \dots a_1)}^{\ \ b} \mathbf{x}_b \quad r = 3, \dots, N. \end{aligned}$$

The equations (1, 11*b*, *c*) follow at once from (1, 12*a*, *b*).

Lemma (1, 4). *If for each* $r = 2, \dots, N$ *and* $q' = 3, \dots, N$ *we have*

(1, 13) *a*) $c_r \neq 0$, (cf. (1, 10)) *b*) $\mathbf{n}_{a_{q'} \dots a_1} y^{a_1} \dots y^{a_p} \neq 0$, $(p = 2, \dots, q')$

then the osculating space $\overset{s}{P}_{m_s}$ [4]) *is spanned by the points* $\mathbf{n}_{a_s \dots a_1}$ $(s = 1, \dots, N;$ $\mathbf{x}_a \equiv \mathbf{n}_a)$.

Proof. The equations (4, 2) in *PI* show that $\overset{s}{P}_{m_s}$ may be thought of as spanned by the points

(1, 14) $$\mathbf{x}, \mathbf{x}_a \equiv \mathbf{n}_a, \dots, \mathbf{n}_{a_{s-1} \dots a_1}, \mathbf{n}_{a_s \dots a_1}.$$

On the other hand $\mathbf{x}$ is a linear combination of $\mathbf{n}_a$, $(N+1)\,\mathbf{x} = y^a \mathbf{n}_a$ and if (1, 13) hold then by virtue of (1, 11) $\mathbf{n}_{a_{q-1} \dots a_1}$ is a linear combination of the points

$$\mathbf{n}_{a_q \dots a_1}, \mathbf{n}_{a_{q-2} \dots a_1}, \dots, \mathbf{n}_a \qquad (q = 2, \dots, N).$$

Hence the space spanned by the points (1, 14) is identical with the space spanned by $\mathbf{n}_{a_s \dots a_1}$.

[3]) For $r = 3$ one has to put $\sum_2^{r-2} {}_s \equiv 0$.

[4]) Cf. § 1 in *PI*.

In the next Lemma we use (1, 8b) and put

$$H_{a_u} \equiv \frac{1}{K} K_{a_{u+1} \ldots a_1}{}^{a_u+1} \, y^{a_{u-1}} \ldots y^{a_1}$$

so that we have

(1, 15) $$y^a H_a = 1.$$

Lemma (1, 5). *The equation*

(1, 16) $$H^c_a (\delta^a_b + y^a H_b) = \delta^c_b$$

admits only one solution H^c_a. *If we put*

(1, 17) $$\begin{cases} a) & H_{ab}^{\;\;c} \equiv 2H_{(a}H^c_{b)} = H_{(ab)}^{\;\;\;\;c} \\ b) & H_{a_{r+1}\ldots a_1}^{\qquad c} \equiv H_{e(a_{r+1}}^{\qquad c} H_{a_r \ldots a_1)}^{\qquad e} = H_{(a_{r+1}\ldots a_1)}^{\qquad\quad c} \end{cases} \qquad (r = 2, \ldots, N-1)$$

then we have

(1, 18) $$a) \;\; y^a H_{ab}^{\;\;c} = \delta^c_b \qquad b) \;\; y^{a_{r+1}} H_{a_{r+1}\ldots a_1}^{\qquad c} = H_{a_r \ldots a_1}^{\qquad c}.$$

Proof. The projective tensor $\delta^a_b + y^a H_b$ (homogeneous of degree 0) has obviously the rank $m+1$. Hence (1, 16) admits only one solution H^c_a. On the other hand we obtain from (1, 17a), (1, 17b) for $r = 2$, (1, 15) and (1, 16),

(1, 19a) $$y^a H_{ab}^{\;\;c} = H^c_b + H_b H^c_a y^a = H^c_a (\delta^a_b + y^a H_b) = \delta^c_b$$

(1, 19b) $$\begin{cases} y^a H_{abc}^{\;\;\;d} = \frac{1}{3} y^a (H_{ea}^{\;\;d} H_{bc}^{\;\;e} + H_{eb}^{\;\;d} H_{ca}^{\;\;e} + H_{ec}^{\;\;d} H_{ab}^{\;\;e}) \\ \qquad = \frac{1}{3} (\delta^d_e H_{bc}^{\;\;e} + H_{eb}^{\;\;d} \delta^e_c + H_{ec}^{\;\;d} \delta^e_b) = H_{bc}^{\;\;d} \end{cases}$$

and these equations prove (1, 18a) as well as (1, 18b) for $r = 2$. The remaining equations (1, 18b) may be obtained by usual induction.

§ 2. *Privileged points.*

Definition (2, 1). *An object Ω with the components* $\Omega_{a_q \ldots a_1}^{\;\ldots b \ldots}$ *will be termed a privileged object if the equation*

(2, 1) $$y^{a_s} \Omega_{a_q \ldots a_1}^{\;\ldots b \ldots} = 0 \qquad (s = 1, \ldots, q)$$

holds and is (x, y, g)-invariant (cf. Definition (1, 2) *in PI).*

Theorem (2, 1a). *Let $N \geq 2$. Then*

(2, 2a) $$\underline{\mathbf{N}_{ab} = \mathbf{N}_{(ab)} = \mathbf{n}_{ab} - c_{21} H_{ab}^{\;\;c} \mathbf{x}_c}$$
$$(c_{21} = c_2)$$

are privileged points,

(2, 3a) $$y^a \mathbf{N}_{ab} = y^a \mathbf{N}_{ba} = 0$$

homogeneous of degree $N - 1$

(2, 4a) $$\dot{\mathbf{N}}_{ab} = g^{(N-1)} \mathbf{N}_{ab}.$$

Proof. We have from (2, 2a) by virtue of (1, 18a) and (1, 11a)

$$y^a \mathbf{N}_{ab} = y^a \mathbf{N}_{(ab)} = (c_2 - c_{21}) \mathbf{x}_b$$

and consequently if we put $c_{21} = c_2$ we have (2, 3a). This equation is obviously (x, y)-invariant. Because $H_{ab}^{\;\;d}$ is homogeneous of degree -1, $\mathbf{x}_c$ homogeneous of degree N and $\mathbf{n}_{ab}$ homogeneous of degree $N-1$, we obtain (2, 4a). Hence (2, 3a) is also g-invariant.

Theorem (2, 1b). *Let* $N \geqq 3$. *Then* (2, 2a) *and*

$$(2, 2b) \qquad \frac{\mathbf{N}_{cba} = \mathbf{N}_{(cba)} = \mathbf{n}_{cba} - c_{32}\, H_{(bc}^{\;\;e}\, \delta_{a)}^{f}\, \mathbf{N}_{ef} - (c_{21}\, c_3\, H_{(cba)}^{\;\;\;\;e} - K_{(cba)}^{\;\;\;\;e})\, \mathbf{x}_e}{c_{32} = \dfrac{3\,c_3}{2}}$$

are privileged points

$$(2, 3b) \qquad y^{a_s}\, \mathbf{N}_{a_3 a_2 a_1} = 0 \qquad s = 1, 2, 3$$

and the points (2, 2b) *are homogeneous of degree* $N-2$

$$(2, 4b) \qquad \dot{\mathbf{N}}_{cba} = g^{(N-2)}\, \mathbf{N}_{cba}.$$

Proof. Let $N \geqq 3$ and consider the equation

$$(2, 5a) \qquad \mathbf{N}_{cba} = \mathbf{n}_{cba} - Q_{cba}^{\;\;\;de}\, \mathbf{N}_{de} - Q_{cba}^{\;\;\;d}\, \mathbf{x}_d$$

where the Q's are to be found. Using (1, 11b) and (2, 2a) we obtain

$$(2, 5b) \qquad \begin{cases} y^c\, \mathbf{N}_{cba} = c_3\, \mathbf{n}_{ba} - y^c\, K_{(cba)}^{\;\;\;\;d}\, \mathbf{x}_d - y^c\, [Q_{cba}^{\;\;\;de}\, \mathbf{N}_{de} + Q_{cba}^{\;\;\;d}\, \mathbf{x}_d] \\ \qquad = (c_3\, \delta_{(ba)}^{de} - y^c\, Q_{cba}^{\;\;\;de})\, \mathbf{N}_{de} + [c_3\, c_{21}\, H_{ba}^{\;\;d} - y^c\, K_{(cba)}^{\;\;\;\;d} - y^c\, Q_{cba}^{\;\;\;d}]\, \mathbf{x}_d. \end{cases}$$

Because

$$y^c\, H_{cba}^{\;\;\;d} = y^c\, H_{(cba)}^{\;\;\;\;d} = H_{ba}^{\;\;d}$$

the tensor

$$(2, 6a) \qquad Q_{cba}^{\;\;\;d} \equiv c_3\, c_{21}\, H_{cba}^{\;\;\;d} - K_{(cba)}^{\;\;\;\;d} = Q_{(cba)}^{\;\;\;\;d}$$

reduces the last member on the right hand side to zero. On the other hand we have by virtue of (2, 3a)

$$3\, y^c\, H_{(cb}^{\;\;d}\, \delta_{a)}^{e}\, \mathbf{N}_{de} = y^c\, (H_{cb}^{\;\;d}\, \delta_a^e + H_{ba}^{\;\;d}\, \delta_c^e + H_{ac}^{\;\;d}\, \delta_b^e)\, \mathbf{N}_{de}$$
$$= (\delta_{ba}^{de} + \delta_{ab}^{de})\, \mathbf{N}_{de} = 2\, \delta_{(ba)}^{de}\, \mathbf{N}_{de}.$$

Hence the tensor

$$(2, 6b) \qquad Q_{cba}^{\;\;\;de} \equiv \frac{3\, c_3}{2}\, H_{(cb}^{\;\;d}\, \delta_{a)}^{e} = Q_{(cba)}^{\;\;\;\;de}$$

reduces the first member on the right hand side of (2, 5b) to zero so that we have $y^c\, \mathbf{N}_{cba} = 0$. This equation together with $\mathbf{N}_{cba} = \mathbf{N}_{(cba)}$ (which we obtain from (2, 5a) and (2, 6)) leads to (2, 3b). The remaining part of the theorem is very easily proved.

Note. If $N > 3$ then

$$y^d\, H_{dcba}^{\;\;\;\;e} = H_{cba}^{\;\;\;e}$$

and by virtue of (1, 8a) *and* (1, 9b) (used for the first time) for $s = 3$, $p = 4$

$$(2, 7) \qquad y^d\, D_{(d}\, K_{cba)}^{\;\;\;\;e} = y^d\, D_d\, K_{(cba)}^{\;\;\;\;e} = -2\,(Q+1)\, K_{(cba)}^{\;\;\;\;e}.$$

Hence $Q_{cba}{}^{d}$ as defined by (2, 6*a*) satisfies the relation

$$Q_{cba}{}^{e} = y^{d}\, P_{dcba}{}^{e} \tag{2, 8a}$$

where

$$P_{dcba}{}^{e} = P_{(dcba)}{}^{e} \equiv c_3\, c_{21}\, H_{dcba}{}^{e} + \frac{1}{2(Q+1)}\, D_{(d} K_{cba)}{}^{e}\,. \tag{2, 8b}$$

In the next section we shall generalize this equation in order to be able to generalize the results of Theorems (2, 1).

§ 3. *Auxiliary Lemma.*

In the following lemma we use the abbreviations

$$(3,\,1)\quad \begin{aligned} &a)\quad H_{a_u \ldots a_s \ldots a_1}{}^{b_s \ldots b_1} \equiv H_{(a_u \ldots a_s}{}^{b_s}\, \delta_{a_{s-1} \ldots a_1)}^{b_{s-1} \ldots b_1} \;{}^{5)} \\ &b)\quad k_{a_{u+1} \ldots a_s \ldots a_1}{}^{b_s \ldots b_1} \equiv \left\{\!\!\left\{ K_{(a_{u+1} \ldots a_s}{}^{b_s}\, \delta_{a_{s-1} \ldots a_1)}^{b_{s-1} \ldots b_1} \right\}\!\!\right\} \\ &c)\quad k_{a_{u+1} \ldots a_1}{}^{b_1} \equiv K_{(a_{u+1} \ldots a_1)}{}^{b_1} \\ &\qquad (u = 2, \ldots, N;\; s = 2, \ldots, u-1) \end{aligned}$$

and

$$(3,\,1d)\quad \left\{ \begin{array}{ll} c_{r\,r-1} \equiv \dfrac{r c_r}{2}\,{}^{6)}, & c'_r \equiv c_r\, k_r, \\ r = 2, \ldots, N & r = 3, \ldots, N \end{array} \right.$$

where $k_3 = 1$ and k_r for $r > 3$ is the number taken from the equation $(r = 4, \ldots N)$

$$\left(k_r\, y^{a_r}\, H_{a_r a_{r-1} a_{r-2} \ldots a_1}{}^{b_{r-2} \ldots b_1} - H_{a_{r-1} a_{r-2} \ldots a_1}{}^{b_{r-2} \ldots b_1} \right) t_{(b_{r-2} \ldots b_1)} = 0$$

which holds for any privileged tensor $t_{(b_{r-2} \ldots b_1)}$ whatsoever. Moreover, if $A_{\ldots}^{\ldots a \ldots}$ and $B_{\ldots}^{\ldots a \ldots}$ are two tensors which satisfy the equation

$$(A_{\ldots}^{\ldots a \ldots} - B_{\ldots}^{\ldots a \ldots})\, t_{\ldots a \ldots} = 0 \tag{3, 1e}$$

for any privileged tensor $t_{\ldots a \ldots}$ whatsoever, then we write

$$A_{\ldots}^{\ldots a \ldots} \cong B_{\ldots}^{\ldots a \ldots}\,. \tag{3, 1f}$$

Lemma (3, 1). *If $N \geqq 4$ and if a set of tensors Q satisfies the following conditions*

$$(3,\,2)\quad \left\{ \begin{aligned} &a)\quad Q_{a_u a_{u-1} \ldots a_1}{}^{c_{u-1} \ldots c_1} \equiv c_{u\,u-1}\, H_{a_u a_{u-1} \ldots a_1}{}^{c_{u-1} \ldots c_1} \qquad u = 2, \ldots, N \\ &b)\quad Q_{a_v a_{v-1} a_{v-2} \ldots a_1}{}^{c_{v-2} \ldots c_1} \equiv c_{v-1\,v-2}\, c'_v H_{a_v a_{v-1} a_{v-2} \ldots a_1}{}^{c_{v-2} \ldots c_1} - k_{a_v a_{v-1} a_{v-2} \ldots a_1}{}^{c_{v-2} \ldots c_1} \qquad v = 3, \ldots, N\;{}^{6)} \\ &c)\quad y^{a_w} Q_{a_w \ldots a_s \ldots a_1}{}^{c_s \ldots c_1} = c_w\, Q_{a_{w-1} \ldots a_s \ldots a_1}{}^{c_s \ldots c_1} \\ &\qquad - y^{a_w} \Big[k_{a_w \ldots a_s \ldots a_1}{}^{c_s \ldots c_1} + \sum_{s+1}^{w-2}{}_{q}\, k_{a_w \ldots a_q \ldots a_1}{}^{b_q \ldots b_1}\, Q_{b_q \ldots b_s \ldots b_1}{}^{c_s \ldots c_1} \Big]\,{}^{6)}, \qquad \begin{array}{l} w = 4, \ldots N \\ s = 1, 2, \ldots, w-3 \end{array} \end{aligned} \right.$$

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[5]) $H_{a_u \ldots a_1}{}^{b_1}$ is defined by (1, 17).

[6]) For c_r cf. (1, 10).

then the equation

$$(3,3)\quad y^{a_w} P_{a_w \ldots a_s \ldots a_1}^{\quad c_s \ldots c_1} \cong Q_{a_{w-1} \ldots a_s \ldots a_1}^{\quad c_s \ldots c_1} \qquad w = 4, \ldots, N;\ s = 1, \ldots, w-3$$

admits a solution $P_{a_w \ldots a_s \ldots a_1}^{\quad c_s \ldots c_1} = P_{(a_w \ldots a_s \ldots a_1)}^{\quad c_s \ldots c_1}$ *homogeneous of degree* $s - w$, *which is a function of the H's, K's as well as of the derivatives up to* $D_{a_w} \ldots D_{a_{q+1}} k_{a_q \ldots a_1}^{\quad c}$ $(q = 3, \ldots, w-1)$ [7]) *and consequently* $Q_{a_w \ldots a_s \ldots a_1}^{\quad c_s \ldots c_1}$ *satisfies the equation*

$$(3,2d)\quad \begin{cases} Q_{a_w \ldots a_s \ldots a_1}^{\quad c_s \ldots c_1} \cong c_w P_{a_w \ldots a_s \ldots a_1}^{\quad c_s \ldots c_1} - [k_{a_w \ldots a_s \ldots a_1}^{\quad c_s \ldots c_1} + \\ + \sum_{s+1}^{w-2} {}_q\, k_{a_w \ldots a_q \ldots a_1}^{\quad b_q \ldots b_1} Q_{a_q \ldots b_s \ldots b_1}^{\quad c_s \ldots c_1}] \qquad w = 4, \ldots, N,\ s = 1, 2, \ldots w-3. \end{cases}$$

Proof. We see from Theorems (2, 1) that the conditions $(3, 2a, b)$ for $u = 2, 3$, $v = 3$ are satisfied [7a]) while $(3, 2c)$ reduces for $w = 4$ to

$$(3.4a)\qquad y^d Q_{dcba}^{\quad e} = c_4 Q_{cba}^{\quad e} - y^d [K_{(dcba)}^{\quad e} + c_{21} k_{dcba}^{\quad ij} H_{ij}^{\ e}]$$

The equation (3, 3) (for $w = 4$) is equivalent to $(2, 8a)$, where P is given by $(2, 8b)$ so that we have

$$(3.5)\quad Q_{dcba}^{\quad e} = c_4 \left[c_3 c_{21} H_{dcba}^{\quad e} + \frac{1}{2(Q+1)} D_{(d} K_{cba)}^{\quad e} \right] - (K_{(dcba)}^{\quad e} + c_{21} k_{dcba}^{\quad ij} H_{ij}^{\ e})$$

which proves our lemma for $w = 4$ (and $s = 1 = w - 3$). For the case $w = 5 \leqq N$ we have to consider $Q_{a_5 \ldots a_1}^{b_s \ldots b_1}$, $s = 1, 2$. The tensor $Q_{a_4 \ldots a_2 a_1}^{\quad b_2 b_1}$ which appears in $(3, 2c)$ for $w = 5$, $s = 2$ is given by $(3, 2b)$ for $v = 4$. Because of (2, 7) and $(1, 9b)$ we have

$$(3.6a)\quad \begin{cases} y^{a_5} D_{a_5} k_{a_4 a_3 a_2 a_1}^{\quad b_2 b_1} = 3\, y^{a_5} D_{a_5} K_{(a_4 a_3 a_2}^{\quad b_2} \delta_{a_1)}^{b_1} = \\ \cong \frac{15}{4}\, y^{a_5} D_{(a_5} K_{a_4 a_3 a_2}^{\quad b_2} \delta_{a_1)}^{b_1} = \frac{5}{4}\, y^{a_5} D_{(a_5} k_{a_4 a_3 a_2 a_1)}^{\quad b_2 b_1} \end{cases}$$

$$(3.6b)\quad 3\, y^{a_5} D_{a_5} K_{(a_4 a_3 a_2}^{\quad b_2} \delta_{a_1)}^{b_1} = -\,3 \,.\, 2\,(1+Q)\, K_{(a_4 a_3 a_2}^{\quad b_2} \delta_{a_1)}^{b_1} = -\,2\,(1+Q)\, k_{a_4 a_3 a_2 a_1}^{\quad b_2 b_1}$$

and

$$(3.6c)\qquad y^{a_5} H_{a_5 a_4 a_3 a_2 a_1}^{\quad b_2 b_1} \cong \tfrac{4}{5} H_{a_4 a_3 a_2 a_1}^{\quad b_2 b_1}.$$

Consequently

$$(3.7a)\qquad Q_{a_5 a_4 a_3 a_2 a_1}^{\quad b_2 b_1} \cong c_5 P_{a_5 a_4 a_3 a_2 a_1}^{\quad b_2 b_1} - (k_{a_5 a_4 a_3 a_2 a_1}^{\quad b_2 b_1} + k_{a_5 a_4 a_3 a_2 a_1}^{\quad b_3 b_2 b_1} Q_{b_3 b_2 b_1}^{\quad c_2 c_1})$$

where

$$(3.7b)\quad P_{a_5 a_4 a_3 a_2 a_1}^{\quad b_2 b_1} = P_{(a_5 a_4 a_3 a_2 a_1)}^{\quad b_2 b_1} = \tfrac{5}{4} \left[c_{32} c_4' H_{a_5 a_4 a_3 a_2 a_1}^{\quad b_2 b_1} + \frac{1}{2(1+Q)} D_{(a_5} k_{a_4 a_3 a_2 a_1)}^{\quad b_2 b_1} \right]$$

and these two equations prove our lemma for $w = 5 \leqq N$ and $s = 2$. In

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[7]) Its construction will be given in the proof.

[7a]) cf. the equations (2, 5) and (2, 6).

order to complete the proof for $w = 5$ and $s = 1$, we use the relationships deduced from (1, 8a) and (1, 9b)

$$y^{a_5} D_{(a_5} D_{a_4} K_{a_3 a_2 a_1)}{}^{b} = y^{a_5} D_{a_5} D_{(a_4} K_{a_3 a_2 a_1)}{}^{b_1} = -3(1+Q) D_{(a_4} K_{a_3 a_2 a_1)}{}^{b_1}$$

$$y^{a_5} D_{(a_5} K_{a_4 a_3 a_2 a_1)}{}^{b_1} = y^{a_5} D_{a_5} K_{(a_4 a_3 a_2 a_1)}{}^{b} = -3(1+Q) K_{(a_4 a_3 a_2 a_1)}{}^{b_1}$$

so that we have by virtue of (3, 5) and (3, 6)

$$Q_{a_4 a_3 a_2 a_1}{}^{b_1} = y^{a_5} P_{a_5 \ldots a_1}{}^{b_1} \tag{3, 8a}$$

where

$$\left\{\begin{aligned} & P_{a_5 \ldots a_1}{}^{c_1} = P_{(a_5 \ldots a_1)}{}^{c_1} \cong c_4 \left[c_3 c_{21} H_{a_5 \ldots a_1}{}^{c_1} - \frac{1}{2.3(1+Q)^2} D_{(a_5} D_{a_4} K_{a_3 a_2 a_1)}{}^{e_1} \right] \\ & \qquad + \left[\frac{1}{3(Q+1)} D_{(a_5} K_{a_4 \ldots a_1)}{}^{b_1} + \frac{5 c_{21}}{8(1+Q)} D_{(a_5} k_{a_4 a_3 a_2 a_1)}{}^{b_2 b_1} H_{c_2 b_1}{}^{b_1} \right] \end{aligned}\right. \tag{3, 8b}$$

and consequently

$$Q_{a_5 \ldots a_1}{}^{c_1} \cong c_5 P_{a_5 \ldots a_1}{}^{c_1} - \left(k_{a_5 \ldots a_1}{}^{c_1} + \sum_{2}^{3}{}_q\, k_{a_w \ldots a_q \ldots a_1}{}^{b_q \ldots b_1} Q_{b_q \ldots b_1}{}^{c_1} \right). \tag{3, 8c}$$

The equations (3, 8) prove the lemma for $w = 5 \leqq N$, $s = 1$. Let us now suppose that we already proved the lemma for all $x = 4, 5, \ldots, w' < N$. Then we have in particular

$$Q_{a_{w'} \ldots a_s \ldots a_1}{}^{c_s \ldots c_1} \cong c_{w'} P_{a_{w'} \ldots a_s \ldots a_1}{}^{c_s \ldots c_1} - \left[k_{a_{w'} \ldots a_s \ldots a_1}{}^{c_s \ldots c_1} + \sum_{s+1}^{w'-2}{}_q\, k_{a_{w'} \ldots a_q \ldots a_1}{}^{b_q \ldots b_1} Q_{b_q \ldots b_s \ldots b_1}{}^{c_s \ldots c_1} \right] \tag{3, 9}$$

$$s = 1, \ldots, w' - 3$$

where $P_{a_{w'} \ldots a_s \ldots a_1}{}^{c_s \ldots c_1} = P_{(a_{w'} \ldots a_s \ldots a_1)}{}^{c_s \ldots c_1}$ is a function of the H's and K's as well as of the derivatives up to $D_{a_{w'}} \ldots D_{a_{q+1}} k_{a_q \ldots a_1}{}^{b_1}$, $(q = 3, \ldots, w'-1)$ and

$$\left\{\begin{aligned} & y^{a_{w'+1}} Q_{a_{w'+1} \ldots a_s \ldots a_1}{}^{c_s \ldots c_1} = c_{w'+1} Q_{a_{w'} \ldots a_s \ldots a_1}{}^{c_s \ldots c_1} \\ & \qquad - y^{a_{w'+1}} \left[k_{a_{w'+1} \ldots a_s \ldots a_1}{}^{c_s \ldots c_1} + \sum_{s+1}^{w'-1}{}_q\, k_{a_{w'} \ldots a_q \ldots a_1}{}^{b_q \ldots b_1} Q_{b_q \ldots b_s \ldots b_1}{}^{c_s \ldots c_1} \right] \\ & \qquad s = 1, 2, \ldots, w' - 2. \end{aligned}\right. \tag{3, 10}$$

Using now the conditions (1, 8a) and (1, 9b) we prove by the same argument as before that

$$Q_{a_{w'} \ldots a_s \ldots a_1}{}^{c_s \ldots c_1} \cong y^{a_{w'+1}} P_{a_{w'+1} \ldots a_s \ldots a_1}{}^{c_s \ldots c_1} \qquad s = 1, \ldots, w' - 3 \tag{3, 11}$$

where $P_{a_{w'+1} \ldots a_s \ldots a_1}{}^{c_s \ldots c_1} = P_{(a_{w'+1} \ldots a_s \ldots a_1)}{}^{c_s \ldots c_1}$ is a function of the H's and k's as well as of the derivatives up to $D_{a_{w'+1}} \ldots D_{a_{q+1}} k_{a_q \ldots a_1}$ $(q = 3, \ldots w')$. Hence we have from (3, 10) and (3, 11) the equation

$$\left\{\begin{aligned} & Q_{a_{w'+1} \ldots a_s \ldots a_1}{}^{b_s \ldots b_1} \cong c_{w'+1} P_{a_{w'+1} \ldots a_s \ldots a_1}{}^{c_s \ldots c_1} - \\ & - \left(k_{a_{w'+1} \ldots a_s \ldots a_1}{}^{c_s \ldots c_1} + \sum_{s+1}^{w'-1}{}_q\, k_{a_{w'+1} \ldots a_q \ldots a_1}{}^{b_q \ldots b_1} Q_{b_q \ldots b_s \ldots b_1}{}^{c_s \ldots c_1} \right) \end{aligned}\right. \tag{3, 12}$$

for $s = 1, \ldots, w' - 3$. The equation (3, 12) for $s = w' - 2$ may be obtained by a similar argument based on (3, 2b) for $v = w'$. The induction based on these results proves our Lemma but for the statement of the homogeneity of the P's. In order to prove this statement we observe first from (3, 2a, b) that $Q_{a_u a_{u-1} \ldots a_1}^{\;\;\;\;\;b_{u-1} \ldots b_1}$ resp. $Q_{a_u \ldots a_{u-2} \ldots a_1}^{\;\;\;\;\;b_{u-2} \ldots b_1}$ is homogeneous of degree -1 resp. -2. Hence from (3, 2c) for $s = w - 3$ we see that $Q_{a_w \ldots a_{w-3} \ldots a_1}^{\;\;\;\;\;b_{w-3} \ldots b_1}$ must be homogeneous of degree -3. Consequently, from the same equation for $w = x - 1$ we obtain by the same argument that $Q_{a_x \ldots a_{x=4} \ldots a_1}^{\;\;\;\;\;b_{x-4} \ldots b_1}$ is homogeneous of degree -4. Proceeding in the same way we arrive at the conclusion that $Q_{a_w \ldots a_s \ldots a_1}^{\;\;\;\;\;b_s \ldots b_1}$ is homogeneous of degree $s - w$, $(w = 4, \ldots, N;\ s = 1, \ldots w - 3)$. Hence $P_{a_w \ldots a_s \ldots a_1}^{\;\;\;\;\;b_s \ldots b_1}$ which satisfies (3, 3) must be by virtue of (3, 2d) and (3, 3) homogeneous of degree $s - w$.

§ 4. *Privileged points. Continuation.*

Lemma (3, 1) enables us to prove the following

Theorem (4, 1). *Let $N \geqq 4$ and let the tensors Q be defined by* (3, 2a,b,c,d). *Then the points*

$$(4,\ 1a)\qquad \begin{cases} \mathbf{N}_{a_r \ldots a_1} = \mathbf{N}_{(a_r \ldots a_1)} \equiv \mathbf{n}_{a_r \ldots a_1} - \sum\limits_{1}^{r-1}{}_s\, Q_{a_r \ldots a_s \ldots a_1}^{\;\;\;\;\;b_s \ldots b_1}\, \mathbf{N}_{b_s \ldots b_1}, \\ (r = 2, \ldots, N;\ \mathbf{N}_a \equiv \mathbf{x}_a) \end{cases}$$

are privileged points

$$(4,\ 1b)\qquad y^{a_p}\, \mathbf{N}_{a_r \ldots a_1} = 0 \qquad p = 1, \ldots, r$$

homogeneous of degree $N + 1 - r$

$$(4,\ 1c)\qquad \dot{\mathbf{N}}_{a_r \ldots a_1} = g^{(N+1-r)}\, \mathbf{N}_{a_r \ldots a_1}.$$

The proof may be accomplished in four steps:

a) Theorems (2, 1) are particular cases of our theorem for $r = 2$ resp. $r = 3$ [8]). Let us assume that we proved our Theorem for all $x = 4, \ldots, r' - 1, r' \leqq N$.

b) Denote by $Q_{a_{r'} \ldots a_s \ldots a_1}^{\;\;\;\;\;b_s \ldots b_1}$ a set of unknown tensors and consider the points

$$(4,\ 2)\qquad \mathbf{N}_{a_{r'} \ldots a_1} \equiv \mathbf{n}_{a_{r'} \ldots a_1} - \sum_{1}^{r'-1}{}_s\, Q_{a_{r'} \ldots a_s \ldots a_1}^{\;\;\;\;\;b_s \ldots b_1}\, \mathbf{N}_{b_s \ldots b_1}.$$

[8]) For $r = 2$ the equations (3, 2, b, c, d) do not exist. For $r = 3$ the equations (3, 2c, d) do not exist (cf. the equation (2, 8a)).

Using Lemma (1, 3) and the equations (4, 1a) (where instead of r we put $x = 2, 3, \ldots, r'-1$) we obtain from (4, 2)

$$
(4, 3a)\quad \left\{
\begin{aligned}
& y^{a_{r'}} \mathbf{N}_{a_{r'}\ldots a_1} \equiv \left(c_{r'}\, \delta^{b_{r'-1}\ldots b_1}_{(a_{r'-1}\ldots a_1)} - y^{a_{r'}} Q_{a_{r'}a_{r'-1}\ldots a_1}^{b_{r'-1}\ldots b_1}\right) \mathbf{N}_{b_{r'-1}\ldots b_1} \\
& + \left[c_{r'} Q_{a_{r'-1}a_{r'-2}\ldots a_1}^{b_{r'-2}\ldots b_1} - y^{a_{r'}} \left(k_{a_{r'}\ldots a_{r'-2}\ldots a_1}^{b_{r'-2}\ldots b_1} + Q_{a_{r'}\ldots a_{r'-2}\ldots a_1}^{b_{r'-2}\ldots b_1}\right)\right] \mathbf{N}_{b_{r'-2}\ldots b} \\
& + \sum_{1}^{r'-3}{}_{s} \left[c_{r'} Q_{a_{r'-1}\ldots a_s\ldots a_1}^{c_s\ldots c_1} - y^{a_{r'}} \left(Q_{a_{r'}\ldots a_s\ldots a_1}^{c_s\ldots c_1} + k_{a_{r'}\ldots a_s\ldots a_1}^{c_s\ldots c_1} + \right.\right. \\
& \left.\left. + \sum_{s+1}^{r'-2}{}_{q}\, k_{a_{r'}\ldots a_q\ldots a_1}^{b_q\ldots b_1} Q_{b_q\ldots b_s\ldots b_1}^{c_s\ldots b_1}\right)\right] \mathbf{N}_{c_s\ldots c_1}.
\end{aligned}
\right.
$$

On the other hand we have according to (3, 1a), (1, 18b) and *by virtue of* (4, 1b) (for $r'-1$ instead of r)

$$
(4, 4a)\quad \left\{
\begin{aligned}
y^{a_{r'}} H_{a_{r'}a_{r'-1}\ldots a_1}^{b_{r'-1}\ldots b_1} \mathbf{N}_{b_{r'-1}\ldots b_1} &= y^{a_{r'}} H_{(a_{r'}a_{r'-1}}^{b_{r'-1}} \delta_{a_{r'-2}\ldots a_1)}^{b_{r'-2}\ldots b_1} \mathbf{N}_{b_{r'-1}\ldots b_1} \\
&= \frac{2}{r'} \delta_{(a_{r'-1}\ldots a_1)}^{b_{r'-1}\ldots b_1} \mathbf{N}_{b_{r'-1}\ldots b_1}
\end{aligned}
\right.
$$

and moreover according to (3, 2a) for $u = r'-1$ [9])

$$
(4, 4b)\quad \left\{
\begin{aligned}
c_{r'} Q_{a_{r'-1}a_{r'-2}\ldots a_1}^{b_{r'-2}\ldots b_1} \mathbf{N}_{b_{r'-2}\ldots b_1} &= c_{r'}\, c_{r'-1\, r'-2} H_{a_{r'-1}a_{r'-2}\ldots a_1}^{b_{r'-2}\ldots b_1} \mathbf{N}_{b_{r'-2}\ldots b_1} \\
&= c'_{r'}\, c_{r'-1\, r'-2}\, y^{a_{r'}} H_{a_{r'}\, a_{r'-1}a_{r'-2}\ldots a_1}^{b_{r'-2}\ldots b_1} \mathbf{N}_{b_{r'-2}\ldots b_1}.
\end{aligned}
\right.
$$

Hence if we impose on $Q_{a_{r'}\ldots a_s\ldots a_1}^{b_s\ldots b_1}$ ($s = 1, \ldots, r'-1$) the conditions (3, 2a,b,c) for $u = v = w = r'$ then we obtain

$$
(4, 3b)\qquad y^{a_{r'}} \mathbf{N}_{a_{r'}\ldots a_1} = 0.
$$

Moreover from (3, 2a,b,c) for $u = v = w = r'$ we obtain (3, 2d). Hence all tensors $Q_{a_{r'}\ldots a_s\ldots a_1}^{b_s\ldots b_1}$ are symmetric in their subscripts so that we have according to (4, 2)

$$
\mathbf{N}_{a_{r'}\ldots a_1} = \mathbf{N}_{(a_{r'}\ldots a_1)}.
$$

This equation together with (4, 3b) leads to

$$
(4, 3c)\qquad y^{a_p} \mathbf{N}_{a_{r'}\ldots a_1} = 0 \qquad p = 1, \ldots, r'.
$$

On the other hand the $Q_{a_{r'}\ldots a_s\ldots a_1}^{b_s\ldots b_1}$ ($s = 1, \ldots, r'-1$) defined by (3, 2a,b,c,d) are homogeneous of degree $s - r'$. Hence $\mathbf{N}_{a_{r'}\ldots a_1}$ are homogeneous of degree $N + 1 - r'$.

[9]) According to our assumption in the section a) of the proof, the equation (3, 2a), resp. (3, 2b), resp. (3, 2c, d) exist for $u = 2, \ldots, r'-1$, resp. $v = 3, \ldots, r'-1$, resp. $w = 4, \ldots, r'-1$.

c) Starting with Theorems (2, 1) and applying the same arguments as in *b*), we easily prove the statements of our theorems for all $x = 4, \ldots$ $\ldots r' - 1$. Hence the assumption of section *a*) is fulfilled.

d) The usual induction based on the assumption of the section *a*) and on the results of section *b*) proves our theorem for $r = 2, \ldots, N$.

§ 5. *Projective normal spaces.*

Definition (5, 1). *The space spanned by the points* $\mathbf{x}$, $\mathbf{N}_{a_r \ldots a_1}$ *will be denoted by* $\overset{r-1}{N}_{n_{r-1}}$ *and referred to as the* $(r-1)$*st projective normal space of our* $\mathfrak{P}_m$, $(r = 2, \ldots, N)$.

Theorem (5, 1). *The normal space* $\overset{r-1}{N}_{n_{r-1}}$ *has the following properties*

a) *It is* (x, y, g)*-invariant.*

b) *It is contained in the osculating space* $\overset{r}{P}_{m_r}$ *and intersects* $\overset{s}{P}_{m_s}$ $(s < r)$ *only in* $\mathbf{x}$.

c) *If*

$$(5, 1) \qquad a) \quad c_2 c_3 \ldots c_r \neq 0, \qquad b) \quad \mathbf{n}_{a_q \ldots a_1} y^{a_1} \ldots y^{a_p} \neq 0, \qquad q = 3, \ldots, r;\ p = 2, \ldots, r$$

then its dimension is

$$(5, 2) \qquad n_{r-1} = m_r - m_{r-1}.$$

Proof. Because the points $\mathbf{x}$, $\mathbf{N}_{a_r \ldots a_1}$ are (x, y)-invariant, $\overset{r-1}{N}_{n_{r-1}}$ is obviously (x, y)-invariant. Because $\mathbf{x}$, $\mathbf{N}_{a_r \ldots a_1}$ are homogeneous (of degree $N + 1$ resp. $N + 1 - r$) the space spanned by them must be g-invariant.

On the other hand using (4, 1*a*) as well as the equations (4, 2) in *PI* we see that $\mathbf{N}_{a_r \ldots a_1}$ may be expressed in the following way

$$(5, 3) \qquad \mathbf{N}_{a_r \ldots a_1} \equiv \mathbf{x}_{a_r \ldots a_1} + \sum_{1}^{r-1} {}_s\, \Omega_{a_r \ldots a_s \ldots a_1}^{b_s \ldots b_1} \mathbf{x}_{b_s \ldots b_1}.$$

Hence all points $\mathbf{N}_{a_r \ldots a_1}$ are in $\overset{r}{P}_{m_r}$ and consequently $\overset{r-1}{N}_{n_{r-1}} \subset \overset{r}{P}_{m_r}$. Moreover if some point $\mathbf{P} \neq \mathbf{x}$ of $\overset{r-1}{N}_{n_{r-1}}$ is in $\overset{q}{P}_{m_q}$ $(q < r)$ then according to (5, 3) it must be a linear combination of points $y^{a_r} \ldots y^{a_{q+1}} \mathbf{N}_{a_r \ldots a_1}$. Because $\mathbf{N}_{a_r \ldots a_1} \equiv \mathbf{N}_{(a_r \ldots a_1)}$ are privileged points, we have

$$y^{a_r} \ldots y^{a_{q+1}} \mathbf{N}_{a_r \ldots a_1} = 0$$

and consequently there is no point $\mathbf{P} \neq \mathbf{x}$ of $\overset{r-1}{N}_{n_{r-1}}$ in $\overset{q}{P}_{m_q}$ $(q < r)$. Consider now the equations

$$(5, 4) \qquad \begin{cases} a) \quad \mathbf{n}_{a_r \ldots a_1} = \mathbf{N}_{a_r \ldots a_1} + \mathbf{M}_{a_r \ldots a_1} \\ b) \quad \mathbf{M}_{a_r \ldots a_1} \equiv \sum_{1}^{r-1} {}_s\, Q_{a_r \ldots a_s \ldots a_1}^{b_s \ldots b_1} \mathbf{N}_{b_s \ldots b_1}, \qquad (\mathbf{N}_a \equiv \mathbf{x}_a) \end{cases}$$

(equivalent with (4, 1a)) and denote by $\overset{r-1}{P'}_{m'_{r-1}} \subset \overset{r-1}{P}_{m_{r-1}}$ the space spanned by the points $\mathbf{x}$, $\mathbf{M}_{a_r \ldots a_1}$. Suppose $m'_{r-1} < m_{r-1}$. Since $\overset{r-1}{N}_{n_{r-1}}$ intersects $\overset{q}{P}_{m_q}$ $(q < r)$ only in $\mathbf{x}$, the space spanned by the points $\mathbf{n}_{a_r \ldots a_1}$, which satisfy (5, 4$a$), can not be $\overset{r}{P}_{m_r}$ for $m'_{r-1} < m_{r-1}$ and this together with (5, 1) contradicts Lemma (1, 4). Because we can not have $m'_{r-1} > m_{r-1}$ and the assumption $m'_{r-1} < m_{r-1}$ is a contradictory one, we must have $m'_{r-1} = m_{r-1}$, so that the space spanned by the points $\mathbf{x}$, $\mathbf{M}_{a_r \ldots a_1}$ is the osculating space $\overset{r-1}{P}_{m_{r-1}}$. Hence we see from (5, 4a) and from the statement b) that (5, 2) holds.

Note: The points $\mathbf{N}_{(a_r \ldots a_1)}$ $(r = 2, \ldots, N)$ which (together with $\mathbf{x}$) span the normal space $\overset{r-1}{N}_{n_{r-1}}$ are linearly dependent even in the maximal case (cf. equation (4, 1b)):

Theorem (5, 2). *In the maximal case the points* $\mathbf{N}_{(a_r \ldots a_1)}$ *are linearly "interdependent", e.g. any of their linear combination which is equal to zero must be built up as a linear combination of* $y^{a_p}\, \mathbf{N}_{a_r \ldots a_1}$ $(p = 1, \ldots, r)$.

Proof. Introduce a special parameter system for which $y^a = \delta^a_o$ at P. Then (5, 3) reduces to

$$(5,\,5) \qquad \mathbf{N}_{(a_r \ldots a_1)} = \mathbf{x}_{(a_r \ldots a_1)} + \sum_{1}^{r-1} {}_{s}\, \Omega_{(a_r \ldots a_s \ldots a_1)}^{\;b_s \ldots b_1}\, \mathbf{x}_{b_s \ldots b_1} \text{ at } P$$
$$(a_1, \ldots, a_r = 1, \ldots, m)$$

while the remaining equations (5, 3) reduce to identity $0 = 0$. In order to prove our theorem, it is sufficient to prove that $\mathbf{N}_{(a_r \ldots a_1)}$ are linearly independent: The points $\mathbf{x}_{(a_r \ldots a_1)}$ span $\overset{r}{P}_{m_r}$ while the points $\mathbf{x}_{(a_r o\, o \ldots o\, a_{s-1} \ldots a_1)}$ span $\overset{s}{P}_{m_s}$. Hence in the maximal case [where the points $\mathbf{x}_{(a_r \ldots a_1)}$ are linearly independent] the points $\mathbf{x}_{(a_r \ldots a_1)}$ are linearly independent and span (together with $\mathbf{x}$) a n_{r-1}-dimensional space [10] $M_{n_{r-1}} \subset \overset{r}{P}_{m_r}$ not contained in $\overset{r-1}{P}_{m_{r-1}}$. Consequently by virtue of (5, 5) the points $\mathbf{N}_{(a_r \ldots a_1)}$ are linearly independent.

Note. Suppose $m = 1$. Because $y^a\, \mathbf{x}_a = (N+1)\, \mathbf{x}$, the points $\mathbf{x}_0$, $\mathbf{x}_1$ are on the tangential line (the first osculating space $\overset{1}{P}_1$) of $\mathfrak{P}_1$ at $\mathbf{x}$. Introduce a parameter system y^a for which $y^a = \delta^a_0$ at $\mathbf{x}$. Then we have in this parameter system

$$0 = y^a\, \mathbf{N}_{ab} = \mathbf{N}_{ob} \text{ at } \mathbf{x}.$$

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[10] $n_{r-1} = \binom{m+r-1}{r} = \binom{m+r}{r} - \binom{m+r-1}{r-1} = m_r - m_{r-1}.$

Hence the set of points $\mathbf{N}_{ab}$ reduces here to $\mathbf{N}_{11}$ and the space $\overset{1}{N}_1$ spanned by $\mathbf{x}$, $\mathbf{N}_{ab}$ is the line which joins the points $\mathbf{x}$ and $\mathbf{N}_{11}$ (the first projective normal). Because $\overset{1}{N}_1$ is y-invariant it does not depend on the choice of parameters. Hence if we chose again an arbitrary parameter system, we obtain the same straight line $\overset{1}{N}_1$ which contains the points $\mathbf{x}$, $\mathbf{N}_{ab}$. It may be easily proved by the same argument that the space $\overset{r-1}{N}_{n_{r-1}}$ is a straight line (the $(r-1)$st projective normal) which contains the points $\mathbf{x}$, $\mathbf{N}_{a_r \ldots a_1}$ $(r = 2, \ldots N)$.

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