MATHEMATICS

ON THE THEORY OF SPHEROIDAL WAVE FUNCTIONS OF ORDER ZERO

BY

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In this paper, the standard form of the differential equation of spheroidal wave functions [1, 2, 3, 4, 5] of order zero is understood to be

$$\Omega_z\, y = (1-z^2)\, y'' - 2zy' + (\lambda + k^2 z^2)\, y = 0, \tag{1}$$

in which λ and k are independent of z. Those values $\lambda = \lambda_\nu(k)$ for which equation (1) admits a solution $y = y_\nu(z;k)$ that is finite at $z = \pm 1$ are called the characteristic values or eigenvalues of the differential equation (1). The corresponding characteristic solutions or eigenfunctions $y_\nu(z;k)$ are spheroidal wave functions (of order zero) of the first kind and of degree ν, where ν runs through the set of non-negative integers.

In the limiting case $k = 0$ equation (1) reduces to the familiar differential equation of Legendre functions, so that $\lambda_\nu(0) = \nu(\nu+1)$ while $y_\nu(z;0)$ is proportional to $P_\nu(z)$.

It is convenient [2, 4] to normalize the eigenfunctions so that their norm is independent of k, viz.

$$\int_{-1}^{1} \{y_\nu(z;k)\}^2\, \mathrm{d}z = \int_{-1}^{1} \{P_\nu(z)\}^2\, \mathrm{d}z = \frac{2}{2\nu+1}. \tag{2}$$

The remaining ambiguity in algebraic sign is removed by the auxiliary condition

$$\lim_{k\to 0} y_\nu(z;k) = P_\nu(z), \tag{3}$$

which in addition to (2) suffices to fix the eigenfunctions completely, since they are uniform functions of k if k^2 is real. As in the analogous case of Mathieu functions [6] this property ceases to hold if k is unrestricted complex [7]. In this paper, however, k^2 will be taken real so that the question of normalization does not lead to difficulties.

It is known that the set of eigenfunctions of the differential equation (1) is identical with the set of eigenfunctions of the integral equation

$$y(z) = \mu \int_{-1}^{1} e^{kzt}\, y(t)\, \mathrm{d}t. \tag{4}$$

To verify this statement [2], let us substitute in (1)

$$y(z) = \int e^{kzt} h(t)\,\mathrm{d}t.$$

Then, by the familiar procedure of differentiation with respect to z and subsequent integration by parts with respect to t, it is easy to see that

$$\Omega_z\{y(z)\} = (1-t^2)\{kz\,h(t) - h'(t)\}\,e^{kzt} + \int e^{kzt}\,\Omega_t\{h(t)\}\,\mathrm{d}t.$$

Now the integral vanishes if $h(t)$ is a characteristic solution of the differential equation (1), at least if the, as yet unspecified, contour of integration lies wholly inside the regularity domain of $h(t)$. If $h(t)$ is an eigenfunction of (1), which implies that $y(t)$ and $y'(t)$ are finite at $t = \pm 1$, then the integrated term vanishes identically in z at the points $t = \pm 1$. Consequently,

$$\int_{-1}^{1} e^{kzt}\,y_\nu(t;k)\,\mathrm{d}t$$

is a solution of (1) corresponding to the eigenvalue $\lambda_\nu(k)$ of (1). This solution is clearly analytic at $z = \pm 1$, so that it is simply proportional to $y_\nu(z;k)$. Therefore, any eigenfunction of (1) is an eigenfunction of (4). The converse is also true, which completes the proof of our statement.

In passing it may be noted that there does not seem to exist a simple relationship between the eigenvalues $\lambda_\nu(k)$ of the differential equation and the eigenvalues $\mu_\nu(k)$ of the integral equation. As a matter of fact, equation (4) becomes nugatory in the limiting case $k = 0$, in the sense that μ tends to infinity if k tends to zero (unless $\nu = 0$). Generally, it can be proved that

$$\mu_\nu(k) = \frac{(2\nu)!\,(2\nu+1)!}{2^{2\nu+1}(\nu!)^3}\,\frac{1}{k^\nu}\,\{1 + O(k^2)\} \qquad (k \to 0), \tag{5}$$

as is seen by ν-times differentiation of (4) followed by application of $k \to 0$, which gives

$$\mu \sim \frac{\mathrm{d}^\nu P_\nu(z)/\mathrm{d}z^\nu}{k^\nu \int_{-1}^{1} t^\nu P_\nu(t)\,\mathrm{d}t}.$$

In the conventional approach to spheroidal wave functions [**1, 2, 3, 4, 5**], the eigenfunctions of equation (1) are expanded in a series of Legendre polynomials, viz.

$$y(z) = \sum_{n=0}^{\infty} a_n P_n(z), \tag{6}$$

where for simplicity in notation we have omitted any explicit indication as to the degree ν and the parameter k of the eigenfunction $y(z)$ and its coefficients a_n, which actually depend on ν and k. The coefficients a_n can be determined by the use of infinite continued fractions, as in the analogous theory of Mathieu functions. At the same time the technique provides a transcendental equation for the eigenvalues $\lambda = \lambda_\nu(k)$. In accordance with (2), the coefficients are normalized so that

$$\sum_{n=0}^{\infty} \frac{a_n^2}{2n+1} = \frac{1}{2\nu+1}, \quad \lim a_\nu = 1 \quad (k \to 0). \tag{7}$$

For details the reader is referred to an earlier paper [**4**]. Let it suffice to mention that the eigenfunctions are entire functions of z, with an irregular singularity at infinity; and they are either even ($a_{2n+1} = 0$) or odd ($a_{2n} = 0$) with respect to z. Thus

$$y_\nu(-z; k) = (-1)^\nu y_\nu(z; k). \tag{8}$$

For a given value of k and a corresponding eigenfunction $y_\nu(z; k)$ with characteristic value $\lambda_\nu(k)$, the complete solution of equation (1) is often required in physical applications. Spheroidal wave functions of the second, third, etc., kinds are necessarily singular at $z = \pm 1$.

It has been suggested [**1**, **8**] that a spheroidal wave function of the second kind would be given by

$$\bar{y}(z) = \sum_{n=0}^{\infty} a_n Q_n(z), \tag{9}$$

in which Q_n is Legendre's function of the second kind and the coefficients a_n are the same as in the expansion (6). In fact, it was argued that these coefficients were to be found simply by application of the familiar recurrence relations for Legendre functions, which were said to be the same for P_n and Q_n. This, however, is erroneous because the two types of Legendre function behave differently in the beginning, that is, for $n = 0$ and $n = 1$. Among others, in deriving (6) the term $n P_{n-1}$ is rightly equated to zero for $n = 0$, but in the analogous case of (9) the corresponding term $n Q_{n-1}$ must be interpreted as being unity for $n = 0$ owing to the fact that Q_n, considered as a function of n, has a simple pole at $n = -1$ with residue 1. As a consequence,

$$\Omega_z\{\bar{y}(z)\} = k^2 a_0 P_1(z) \quad \text{or} \quad \tfrac{1}{3} k^2 a_1 P_0(z),$$

depending on whether the relevant eigenfunction (6) is even or odd in z. Thus the function $\bar{y}(z)$ does not solve equation (1) if $k \neq 0$, since it may be shown that a_0 and a_1 are never zero for the even and odd eigenfunctions of (1) unless $k = 0$. This peculiarity has been overlooked by Hanson [8] who apparently committed the error for the first time, though the correct expansion was known long before the publication of his paper. The error mentioned has been noted by Stratton and coworkers [3] and by the author [2] almost simultaneously.

According to Herzfeld [9], the series (9) should be supplemented by one of the type (6), viz.

$$Y(z) = \sum_{n=0}^{\infty} a_n Q_n(z) + \sum_{n=0}^{\infty} b_n P_n(z), \tag{10}$$

in which the coefficients b_n satisfy a similar three-term recurrence relation as do the coefficients a_n. From the conditions $b_{n+2}/b_n \sim k^2/4n^2$ $(n \to \infty)$ and

$$\Omega_z \{ \sum_{n=0}^{\infty} b_n P_n(z) \} = -k^2 a_0 P_1(z) \quad \text{or} \quad -\tfrac{1}{3} k^2 a_1 P_0(z)$$

for $\lambda = \lambda_\nu(k)$, the coefficients b_n are uniquely determined. In particular, the link between the two systems of coefficients is provided by an infinite continued fraction for the ratio of the first non-vanishing coefficients a and b. For an even eigenfunction $(a_{2n+1} = 0)$ the additional series $\sum b_n P_n$ is odd in z $(b_{2n} = 0)$, and we have

$$\frac{b_1}{a_0} = \frac{k^2 |}{|\alpha_1 - \lambda} - \frac{\beta_3 |}{|\alpha_3 - \lambda} - \frac{\beta_5 |}{|\alpha_5 - \lambda} - \dots, \tag{11}$$

where λ is the corresponding eigenvalue of the even function, while

$$\alpha_n = n(n+1) - \tfrac{1}{2} k^2 \left\{ 1 + \frac{1}{(2n-1)(2n+3)} \right\},$$

$$\beta_n = \frac{n^2 (n-1)^2}{(2n-1)^2} \frac{k^4}{(2n-3)(2n+1)}.$$

On the other hand, for an odd eigenfuntion $(a_{2n} = 0)$, the series $\sum b_n P_n$ is even in z $(b_{2n+1} = 0)$ and

$$\frac{b_0}{a_1} = \frac{\tfrac{1}{3} k^2 |}{|\alpha_0 - \lambda} - \frac{\beta_2 |}{|\alpha_2 - \lambda} - \frac{\beta_4 |}{|\alpha_4 - \lambda} - \dots, \tag{12}$$

where λ now is the eigenvalue of the odd function.

It may be remarked that the right-hand member of (11), where λ is meant to denote an eigenvalue belonging to an *even* eigenfunction, becomes infinitely large for λ equal to any of the eigenvalues of the *odd* eigenfunctions. The reverse applies to (12). This should be so, since in either case we must take all coefficients a_n identically zero, to have finite values for b_n, so that then the series (10) reduces to one of the type (6), that is, represents an eigenfunction itself. In other words, the "even eigenvalues" are the roots of

$$0 = \alpha_0 - \lambda - \frac{\beta_2 |}{|\alpha_2 - \lambda} - \frac{\beta_4 |}{|\alpha_4 - \lambda} - \dots,$$

and the "odd eigenvalues" are the roots of

$$0 = \alpha_1 - \lambda - \frac{\beta_3 |}{|\alpha_3 - \lambda} - \frac{\beta_5 |}{|\alpha_5 - \lambda} - \dots,$$

in conformity with known results [2, 4].

As follows from their construction, the series $\sum a_n P_n$ and $\sum b_n P_n$ both converge for any finite value of z, and in fact they converge to an entire function of z. On the other hand, the series $\sum a_n Q_n$ diverges at $z = \pm 1$, where it leads to a logarithmic singularity of $Y(z)$ because the individual terms are singular themselves. Since the singular part of

$Q_n(z)$ is $Q_0(z)\,P_n(z)$, it may be anticipated that $Y(z)-Q_0(z)\sum a_n P_n(z)$ remains finite at $z=\pm 1$. As a matter of fact, the function

$$Y(z)-\tfrac{1}{2}\,y(z)\log\frac{z+1}{z-1}$$

is an entire function of z, as will be seen later on.

In order to make $Y(z)$ one-valued, it is convenient to introduce a cut in the complex plane of z in the same way as is usually done for $Q_n(z)$. This cut is the straight line segment drawn from $z=-1$ to $z=1$ along the real axis. For real values of z between $z=-1$ and $z=1$ a spheroidal function of the second kind can be defined in a similar manner as this is done for Q_n in the theory of Legendre functions.

A disadvantage in accepting the expansions (6) and (10) as a canonic system of spheroidal wave functions of order zero is that they require two different sets of coefficients, a_n and b_n, though it is true that their respective recurrence relations are closely related to each other. In this connection it may be emphasized that Stratton and co-workers [3] derive a_n and b_n, as limiting cases, from one and the same difference equation, valid for spheroidal wave functions of unrestricted degree and order, though at the cost of simplicity in presentation.

From the point of view of numerical computation as well as of analysis, it would be profitable to have a single set of coefficients. In fact, such types of solution have been known for a long time. They may conveniently be obtained in the following manner. If in the integral equation (4) we substitute for $y(t)$ the corresponding expansion (6) we get, for all finite values of z,

$$y\,(z)=2\,\mu\sqrt{\frac{\pi}{2}}\,(kz)^{-\frac{1}{2}}\sum_{n=0}^{\infty}a_n\,I_{n+\frac{1}{2}}(kz), \tag{13}$$

where I denotes the modified Bessel function of the first kind. In deriving (13) we have used the known identity

$$\int_{-1}^{1}e^{kzt}\,P_n\,(t)\,\mathrm{d}t=\sqrt{\frac{2\pi}{kz}}\,I_{n+\frac{1}{2}}\,(kz). \tag{14}$$

An alternative way to derive the solution (13) is by direct substitution in equation (1) and employing the recurrence relations for Bessel functions. It then appears that the resulting difference equation for the coefficients in (13) is precisely that for the coefficients in the expansion (6) in Legendre functions. However, equation (13) gives more in that it fixes the constant of proportion in terms of μ, the eigenvalue of the integral equation (4), which can be calculated as soon as the coefficients a_n are known. For example, if the eigenfunction under consideration is even in z (i.e. if ν is even), we substitute $z=0$ in (4), so as to obtain

$$\mu_\nu=\frac{y\,(0)}{2\,a_0}=\frac{1}{2\,a_0}\sum_{n=0}^{\infty}(-1)^n\,\frac{\Gamma\,(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma\,(n+1)}\,a_{2n}\,,\ (\nu\text{ even}). \tag{15}$$

Similarly, for the odd eigenfunctions,

$$\mu_\nu = \frac{3y'(0)}{2ka_1} = \frac{3}{ka_1} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\frac{3}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+1)}\, a_{2n+1}\,, \; (\nu \text{ odd}), \tag{16}$$

which is found by first differentiating (4) with respect to z and then taking z equal to zero.

Now, unlike the Legendre function Q_n, the modified Bessel function of the second kind, $K_{n+\frac{1}{2}}$, is an analytic function of n for all n. Further, $K_{n+\frac{1}{2}}$ and $I_{n+\frac{1}{2}}$ obey the same set of recurrence relations. It is thus evident that a second solution of equation (1) is given by

$$Y^*(z) = \frac{k}{2\mu} \sqrt{\frac{2}{\pi}}\, (kz)^{-\frac{1}{2}} \sum_{n=0}^{\infty} a_n\, K_{n+\frac{1}{2}}(kz), \tag{17}$$

in so far as this series is convergent. This in fact is true if $|z| > 1$, as may be inferred from the asymptotic behaviour of the general term of (17) as $n \to \infty$.

This asymptotic behaviour follows at once from the recurrence relations for a_n and $K_{n+\frac{1}{2}}$, viz.

$$\frac{a_{n+2}}{a_n} \sim \frac{k^2}{4n^2}\,, \quad \frac{K_{n+5/2}(kz)}{K_{n+1/2}(kz)} \sim \frac{4n^2}{k^2 z^2}\,(n \to \infty).$$

Therefore, the ratio of successive terms of (17) tends to z^{-2} so that (17) is absolutely convergent if $|z| > 1$ and divergent if $|z| < 1$. Furthermore, the degree of convergence of (17) for $|z| > 1$ is comparable to that of the series $\sum_n z^{-2n}$, which implies that (17) is useful only if $|z|$ is much greater than unity.

By a similar reasoning it may be shown that the degree of convergence of the analogous series (13) is comparable to that of the series

$$\sum_n (k^2 z/16)^{2n}/(n!)^4,$$

so that (13), besides being convergent for arbitrary z, is much stronger convergent than (17).

In this connection it may be mentioned that the degree of convergence of the expansions (6) and (10) is intermediate between those of (13) and (17). For points outside the cut $(-1 \leqslant z \leqslant 1)$ the former series converge like

$$\sum_n \{\tfrac{1}{4} k\,(z \pm \sqrt{z^2-1})\}^{2n}/(n!)^2.$$

The divergence inside the unit circle of the series (17) for the spheroidal wave function of the third kind, $Y^*(z)$, is a serious disadvantage in physical applications as, e.g., the diffraction of a plane wave by a circular disk or aperture [2]. On the other hand the asymptotic behaviour as $|z| \to \infty$ of (17) is easily recognized (see below) in contradistinction to the case of (10), and it is the function of the third kind $Y^*(z)$ which is required for the representation of travelling waves diverging from the origin of coordinates.

Therefore we have to continue the series expansion (17) analytically inside the unit circle. This can be accomplished as follows [2]. The analogue of (14) is, if $\mathrm{Re}(kz) > 0$,

$$\int_1^\infty e^{-kzt} P_n(t)\, \mathrm{d}t = \sqrt{\frac{2}{\pi kz}}\, K_{n+\frac{1}{2}}(kz). \tag{18}$$

Assume for a while that $k > 0$ and $z > 1$. Then we may substitute equation (18) in (17) and change the order of summation and integration. Hence, by using (6),

$$Y^*(z) = \frac{k}{2\mu} \int_1^\infty e^{-kzt}\, y(t)\, \mathrm{d}t. \tag{19}$$

Now, according to the integral equation (4),

$$\frac{1}{\mu} y(t) = \int_{-1}^{1} e^{kt\tau}\, y(\tau)\, \mathrm{d}\tau,$$

which after substitution in (19) and reversing the order of integration yields

$$Y^*(z) = \tfrac{1}{2} k \int_{-1}^{1} y(\tau)\, \mathrm{d}\tau \int_1^\infty e^{-kt(z-\tau)}\, \mathrm{d}t.$$

The second integral is elementary, so that the final result becomes

$$Y^*(z) = \tfrac{1}{2} e^{-kz} \int_{-1}^{1} \frac{e^{kt} y(t)}{z-t}\, \mathrm{d}t. \tag{20}$$

This identity has been proved for $z > 1$ and $k > 0$. By the principle of analytic continuation it is obvious that the integral (20) continues the function (17) analytically for all values of z outside the cut $(-1 \leqslant \leqslant z \leqslant 1)$. In addition, the restriction $k > 0$ made during the proof is immaterial for the general validity of (20) for all values of k. For a different proof of (20), see [2].

In passing it may be mentioned that the character of the singularities of $Y^*(z)$ follows at once from (20). Obviously,

$$Y^*(z) - \tfrac{1}{2} y(z) \log \frac{z+1}{z-1} = \tfrac{1}{2} \int_{-1}^{1} \frac{e^{-k(z-t)} y(t) - y(z)}{z-t}\, \mathrm{d}t,$$

in which the remaining integral is an analytic function of z for all z.

Clearly, the function $Y(z)$ is a linear combination of the functions $Y^*(z)$ and $y(z)$. As a matter of fact it will be shown that

$$Y(z) - Y^*(z) = (-1)^\nu (k/4\mu^2)\, y(z), \tag{21}$$

which implies that also

$$Y(z) - \tfrac{1}{2} y(z) \log \frac{z+1}{z-1}$$

is an analytic function of z for all finite z, as stated before.

To prove (21) we make use of the following integral, valid if $\mathrm{Re}(z) > 1$:

$$Q_n(z) = \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-zt} t^{-\frac{1}{2}} I_{n+\frac{1}{2}}(t)\, \mathrm{d}t. \tag{22}$$

In view of (13) we thus find that

$$\sum_{n=0}^\infty a_n Q_n(z) = \frac{k}{2\mu} \int_0^\infty e^{-kzt} y(t)\, \mathrm{d}t, \tag{23}$$

valid, e.g., if $z > 1$ and $k > 0$ (see below). Further, by using (19), we arrive at

$$Y^*(z) = \sum_{n=0}^\infty a_n Q_n(z) - \frac{k}{2\mu} \int_0^1 e^{-kzt} y(t)\, \mathrm{d}t, \tag{24}$$

which apparently holds for unrestricted complex values of z outside the cut (the integral in (24) is an entire function of z). For a different proof of (24), see [2]. Now, if equation (24) is compared with equation (10), in which $\sum b_n P_n(z)$ is an entire function of z, it is evident that $Y(z) - Y^*(z)$ is an entire function of z, too. Since, moreover, this difference must be a solution of the differential equation (1) for the same eigenvalue, it is simply proportional to $y(z)$. Thus

$$\begin{cases} [Y(z) - Y^*(z) =] \\ \dfrac{k}{2\mu} \displaystyle\int_0^1 e^{-kzt} y(t)\, \mathrm{d}t + \sum_{n=0}^\infty b_n P_n(z) = a\, y(z), \end{cases} \tag{25}$$

in which a does not depend on z. Let us now apply to (25) the substitution $z \to -z$. If ν is, as before, the degree of the eigenfunction $y(z)$ one has, in addition to (8), that

$$\sum_{n=0}^\infty b_n P_n(-z) = (-1)^{\nu+1} \sum_{n=0}^\infty b_n P_n(z),$$

so that

$$\frac{k}{2\mu} \int_0^1 e^{kzt} y(t)\, \mathrm{d}t + (-1)^{\nu+1} \sum_{n=0}^\infty b_n P_n(z) = (-1)^\nu a\, y(z).$$

By multiplication of this identity by $(-1)^\nu$ and subsequent addition to (25), the series $\sum b_n P_n$ is eliminated and we are left with

$$2a\, y(z) = \frac{k}{2\mu} \left[\int_0^1 e^{-kzt} y(t)\, \mathrm{d}t + (-1)^\nu \int_0^1 e^{kzt} y(t)\, \mathrm{d}t\right]$$

$$= \frac{k}{2\mu} (-1)^\nu \int_{-1}^1 e^{kzt} y(t)\, \mathrm{d}t = (-1)^\nu \frac{k}{2\mu^2}\, y(z),$$

which leads to $a = (-1)^\nu k/4\mu^2$. This completes the proof of (21).

It will be obvious that neither of the series (6) and (10) is convenient for an investigation of the functions $y(z)$ and $Y(z)$ for large values of z. On the other hand, the asymptotic behaviour of $Y^*(z)$ readily follows from equation (20). To that end, let us suppose that $|z| > 1$. Then the integrand of (20) can be expanded in a series uniformly convergent with regard to t $(-1 \leqslant t \leqslant 1)$, so that

$$Y^*(z) = \tfrac{1}{2} e^{-kz} \sum_{n=0}^{\infty} z^{-n-1} \int_{-1}^{1} e^{kt} y(t) t^n \, dt.$$

The remaining integral is expressible in terms of the nth derivative, $y^{(n)}(1)$, of $y(z)$ at $z = 1$, as follows from the integral equation (4) by differentiation with respect to z and substitution of $z = 1$. We thus obtain for the function of the third kind:

$$Y^*(z) = \frac{k}{2\mu} e^{-kz} \sum_{n=0}^{\infty} \frac{y^{(n)}(1)}{(kz)^{n+1}}, \tag{26}$$

which is *absolutely convergent* when $|z| > 1$. The coefficients $y^{(n)}(1)$ in this expansion satisfy a third-order difference equation, viz.

$$\left\{ \begin{aligned} &2(n+1)\, y^{(n+1)}(1) + [n(n+1) - \lambda - k^2]\, y^{(n)}(1) - 2nk^2\, y^{(n-1)}(1) \\ &\qquad\qquad\qquad\qquad - n(n-1)\, k^2\, y^{(n-2)}(1) = 0, \end{aligned} \right. \tag{27}$$

which is most easily obtained by differentiating equation (1) n times and substituting $z = 1$. This difference equation cannot be solved by the technique of continued fractions. It can be verified, however, that (27) admits a solution that makes the expansion (26) convergent outside the unit circle. For numerical purposes the successive derivatives of $y(z)$ at $z = 1$ can be computed from

$$y^{(n)}(1) = \frac{1}{2^n n!} \sum_{m=n}^{\infty} \frac{(m+n)!}{(m-n)!} a_m, \tag{28}$$

which is easy to derive from (6).

It should be noted that the expansion (26) is also obtained when the integral in equation (19) is evaluated by repeated integration by parts. A third way of deriving (26) is to replace the functions $K_{n+\frac{1}{2}}$ in the series (17) by their explicit expressions, which are elementary functions, and then arrange the terms according to powers of z^{-1}.

The last-mentioned procedure can also be applied to the Bessel functions $I_{n+\frac{1}{2}}$ in equation (13), and for the function of the first kind it is then found that

$$y(z) = \mu \left[e^{kz} \sum_{n=0}^{\infty} \frac{(-1)^n y^{(n)}(1)}{(kz)^{n+1}} + (-1)^{\nu+1} e^{-kz} \sum_{n=0}^{\infty} \frac{y^{(n)}(1)}{(kz)^{n+1}} \right], \tag{29}$$

which also follows from repeated integration by parts of equation (4). Either series in (29) converges absolutely when $|z| > 1$, and is divergent inside the unit circle. On the unit circle they have two singularities at $z = \pm 1$. These singularities, however, cancel in the sum in accordance with the fact that $y(z)$ is everywhere analytic.

The order of growth of the eigenfunctions as $|z| \to \infty$ is immediately given by (29). In particular, for positive values of t,

$$y(t) = o\{\exp(|\operatorname{Re} k|\, t)\} \qquad (t \to \infty),$$

so that the integral representations (19) and (23) are valid at least if $|\mathrm{Re}(k)| < \mathrm{Re}\,(kz)$.

An expansion for $Y(z)$ analogous to (26) and (29) is obtained from (21) by simple substitution; thus, for the function of the second kind,

$$Y(z) = \frac{k}{4\mu}\left[e^{-kz}\sum_{n=0}^{\infty}\frac{y^{(n)}(1)}{(kz)^{n+1}} + (-1)^{\nu}\, e^{kz}\sum_{n=0}^{\infty}\frac{(-1)^{n}\,y^{(n)}(1)}{(kz)^{n+1}}\right], \tag{30}$$

valid outside the unit circle.

There exist many other interesting relations between the spheroidal wave functions y, Y and Y^*. For example it follows from (26) and (30) that

$$Y(z) = \tfrac{1}{2}\{Y^*(z) + (-1)^{\nu+1}\, Y^*(-z)\}, \tag{31}$$

and from (26) and (29) that

$$y(z) = -\frac{2\mu^2}{k}\{Y^*(-z) + (-1)^{\nu}\, Y^*(z)\}, \tag{32}$$

which indicate that $Y^*(z)$ and $Y^*(-z)$ form a fundamental system of spheroidal wave functions. Moreover, if in equation (31) the functions Y^* are replaced by their respective integral representations according to (20), we obtain an integral representation for the function of the second kind, viz.

$$Y(z) = \tfrac{1}{2}\int_{-1}^{1}\frac{\cosh(kz-kt)}{z-t}\, y(t)\, \mathrm{d}t, \tag{33}$$

valid for all values of z outside the cut $-1 \leqslant z \leqslant 1$.

Similarly, the identity (32) leads to a new integral equation for the functions of the first kind, viz.

$$y(z) = \frac{2\mu^2}{k}\int_{-1}^{1}\frac{\sinh(kz+kt)}{z+t}\, y(t)\, \mathrm{d}t, \tag{34}$$

which, however, may be obtained directly from (4) by iteration.

In addition to the expansions (6), (10), (13) and (17), there exist numerous other types of solution of the differential equation (1) with a three-term recurrence relation between the coefficients, which thus are open to numerical computations by the technique of continued fractions. As already mentioned, a disadvantage of the set formed by (6) and (10) is that they require two different systems of coefficients, a_n and b_n. Though this does not hold for the series (13) and (17), the latter fails to represent the function of the third kind inside the unit circle.

Some years ago the author succeeded in deriving a complete solution of (1) that possesses all desired features simultaneously. That is to say, it is possible to choose two linearly independent series solutions of the differential equation (1) that are expressible in terms of a single set of coefficients and converge for all values of z outside the cut. This new system of spheroidal wave functions will be now discussed.

According to Baber and Hassé [10], the eigenfunctions of equation (1) can be expanded as follows:

$$y(z) = e^{-kz} \sum_{n=0}^{\infty} c_n P_n(z), \tag{35}$$

in which c_n is a solution of the difference equation

$$\frac{2k(n+1)^2}{2n+3} c_{n+1} + \{n(n+1) - \lambda - k^2\} c_n - \frac{2kn^2}{2n-1} c_{n-1} = 0, \tag{36}$$

satisfying the boundary conditions

$$c_n = 0 \quad (n < 0), \qquad c_{n+1}/c_n \sim k/n \quad (n \to \infty). \tag{37}$$

Except for a constant of proportion, the solution c_n is unique for any fixed characteristic value λ. This constant is uniquely determined if the normalization (2) is assumed.

The characteristic values now correspond to the roots of

$$0 = -\Lambda + \frac{p_0 |}{|2-\Lambda} + \frac{p_1 |}{|6-\Lambda} + \ldots + \frac{p_{n-1} |}{|n(n+1)-\Lambda} + \ldots, \tag{38}$$

in which

$$\Lambda = \lambda + k^2 \quad , \quad p_n = \frac{(4n+1)^4}{(2n+1)(2n+3)} k^2. \tag{39}$$

As in the former expansions, the coefficients c_n are readily calculated by a process of iteration. For example, in the direction of decreasing values of n, the ratios of successive coefficients follow from

$$\frac{c_n}{c_{n-1}} = \left[\frac{A_n}{k} \{n(n+1) - \Lambda\} + B_n \frac{c_{n+1}}{c_n}\right]^{-1}, \tag{40}$$

in which

$$A_n = (2n-1)/2n^2 \quad , \quad B_n = (n+1)^2 (2n-1)/n^2 (2n+3). \tag{41}$$

Using the asymptotic behaviour of the coefficients c_n as $n \to \infty$ it is further seen that the degree of convergence of (35) is comparable to that of the series

$$\sum_n \{k(z + \sqrt{z^2-1})\}^n / n!$$

if z is outside the cut. Hence, the expansion (35) is not as rapidly convergent as the series (6).

A further disadvantage is that (35) does not indicate whether the corresponding eigenfunction is even or odd in z. This property, of course, is implicit in the coefficients c_n. In virtue of (8) it necessarily follows that also

$$y(z) = (-1)^\nu e^{kz} \sum_{n=0}^{\infty} (-1)^n c_n P_n(z). \tag{42}$$

This symmetry property induces many interesting relations between the coefficients c_n; for example,

$$\frac{\sum_0^\infty c_{2n}}{\sum_0^\infty c_{2n+1}} = \begin{cases} \coth k & (\nu \text{ even}), \\ \tanh k & (\nu \text{ odd}), \end{cases} \tag{43}$$

which easily follows from (35) and (42) upon the substitution $z = 1$. In particular, the sum of the coefficients c_n with even (odd) subscripts is zero for $k = \frac{1}{2} m \pi i$, where m is an odd (even) integer if ν is even and an even (odd) integer if ν is odd.

According to Meixner [7], the same symmetry property is useful in the actual normalization of the coefficients c_n. In fact one has

$$\begin{aligned} \int_{-1}^{1} \{y(z)\}^2 \, dz &= \int_{-1}^{1} \{e^{-kz} \sum c_n P_n(z)\} \{(-1)^\nu e^{kz} \sum (-1)^n c_n P_n(z)\} \, dz \\ &= (-1)^\nu \int_{-1}^{1} \{\sum c_n P_n(z)\} \{\sum (-1)^n c_n P_n(z)\} \, dz \\ &= (-1)^\nu \sum (-1)^n \frac{2}{2n+1} c_n^2. \end{aligned}$$

In other words, the coefficients c_n of (35) and a_n of (6) are to be normalized in essentially the same way. In accordance with (2) and (7) we thus require for the νth eigenfunction

$$\frac{1}{2\nu+1} = \sum_{n=0}^{\infty} (-1)^{n+\nu} \frac{c_n^2}{2n+1}, \quad \lim c_\nu = 1 \ (k \to 0). \tag{44}$$

Let us now substitute Baber and Hassé's series (35) in the integrand of (20). If use is made of Neumann's integral for the Legendre function of the second kind, viz.

$$Q_n(z) = \tfrac{1}{2} \int_{-1}^{1} \frac{P_n(t)}{z-t} \, dt,$$

we obtain the simple result

$$Y^*(z) = e^{-kz} \sum_{n=0}^{\infty} c_n Q_n(z). \tag{45}$$

Therefore, the series (35) and (45) represent a fundamental system of spheroidal wave functions. They are expressed in terms of a single set of coefficients c_n satisfying a three-term recurrence relation, and either series converges in the whole regularity domain of the corresponding function. As emphasized before, it is the function of the third kind (45) which is required for the construction of travelling waves in physical problems. For example, in the problem of diffraction of a plane wave by a circular disk or aperture [2] the function $Y^*(z)$ is required for $k > 0$ and $z = i\zeta$ $(0 \leqslant \zeta < \infty)$, which range of values is completely covered by the uniform representation (45).

At first sight it is surprising that the series (35) and (45) are both solutions of the differential equation (1), if it is remembered that the functions (6) and (9) do not both solve this equation. Of course, by direct substitution of the series (45) in (1), and using with care the recurrence relations for the Q-function, it can be verified independently that the series (45) does solve the differential equation. The difference with the case of (9) is that in the beginning we have to evaluate $n^2 Q_{n-1}$, instead of nQ_{n-1}, when $n = 0$. On account of the extra factor n we may take $n^2 Q_{n-1}$ equal to zero for $n = 0$, like $n^2 P_{n-1}$. In terms of the difference equations underlying the coefficients c_n and a_n, it may be noted that in (36) the factor of c_{n-1} shows a double zero at $n = 0$, while in the corresponding difference equation for the system a_n [4, eq. 3] the factor of a_{n-2} has a single zero at $n = 0$ and a single zero at $n = 1$.

The eigenvalues μ of the integral equation (4) are expressible in terms of c_n in a very simple way. To that end we take $z = 1$ in equation (4) and substitute for $y(t)$ the corresponding series (35). Thus

$$\mu = \frac{y(1)}{2c_0} = \frac{1}{2c_0} e^{-k} \sum_{n=0}^{\infty} c_n. \tag{46}$$

A further question of interest is how the set of coefficients a_n can be expressed in terms of the set c_n and vice versa. The answer is provided by the transformations

$$c_n = (n + \tfrac{1}{2}) \sum_{m=0}^{\infty} A_{n,m}\, a_m, \tag{47}$$

$$a_n = (n + \tfrac{1}{2}) \sum_{m=0}^{\infty} (-1)^{\nu+m} A_{n,m}\, c_m, \tag{48}$$

in which the coefficients $A_{n,m}$ depend only on k, not on the degree ν of the eigenfunction, viz.

$$A_{n,m} = \int_{-1}^{1} e^{kt} P_n(t)\, P_m(t)\, dt. \tag{49}$$

To prove (48), we observe that in view of (6) one has

$$a_n = (n + \tfrac{1}{2}) \int_{-1}^{1} y(t)\, P_n(t)\, dt,$$

which at once leads to (48) and (49) if $y(t)$ is replaced by the corresponding expansion (42). Similarly (47) follows from (35) and (6). The coefficient $A_{n,m}$ is an elementary function of k. It can be shown that

$$A_{n,m} = \sqrt{\frac{2}{\pi k}} \sum_{s=0}^{\min(n,m)} \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)} \frac{\Gamma(n-s+\frac{1}{2})}{\Gamma(n-s+1)} \frac{\Gamma(m-s+\frac{1}{2})}{\Gamma(m-s+1)} \frac{\Gamma(n+m-s+1)}{\Gamma(n+m-s+\frac{1}{2})} I_{n+m-2s+\frac{1}{2}}(k). \tag{50}$$

In concluding this paper, the author wants to emphasize that a similar canonic system as (35) and (45) is possible for spheroidal wave functions of non-negative integral order. Meixner [7] has generalized them so as to be applicable to spheroidal wave functions of unrestricted complex order and degree.

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