MATHEMATICS

AN ARITHMETICAL PROPERTY OF SOME SUMMABLE FUNCTIONS

BY

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§ 1. *Introduction.*

1. 1. If x denotes an irrational number, it is well known, that the sequence

$$x,\ 2x,\ 3x, \ldots$$

is uniformly distributed mod 1; from this it follows that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(nx) = \int_0^1 g(t)\,dt, \tag{1}$$

if $g(t)$ denotes any bounded R-integrable function with period 1.

It is clear that (1) generally becomes false, if in stead of supposing that $g(t)$ is Riemann-integrable, we only assume that $g(t)$ is Lebesgue-integrable, since we can arbitrarily change the value of $g(t)$ at all points nx (mod 1), without changing the integral.

In 1922 A. KHINTCHINE [1]), considering the special case that the periodic function $g(t)$ for $0 \leqq t < 1$ denotes the characteristic function of a measurable set $\mathcal{E}$ in (0, 1), introduced the question whether in this case (1) is true for almost all x. He proved this to be the fact under certain conditions as to the nature of $\mathcal{E}$. One may generalise KHINTCHINE's problem by replacing the sequence (nx) by $(\lambda_n x)$, where

$$\lambda_1 < \lambda_2 < \ldots \tag{2}$$

denotes a sequence of increasing integers. In the special case $\lambda_n = a^n$, where a is a fixed integer $\geqq 2$, RAIKOFF proved that the formula

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(\lambda_n x) = \int_0^1 g(t)\,dt \tag{3}$$

holds almost everywhere in x for any (periodic) function which is Lebesgue-integrable. [2])

F. RIESS [3]) pointed out that RAIKOFF's theorem is actually an instance

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[1]) A. KHINTCHINE, Ein Satz über Kettenbrüche mit arithmetischen Anwendungen. Math. Zeitschr. 18, 289—306 (1923).

[2]) D. RAIKOFF, On some arithmetical properties of summable functions. Rec. Math. Moscou, N.s. 1, 377—384 (1936) (Russian with engl. summary).

[3]) F. RIESS, Sur la théorie ergodique. Comment. Math. Helvet. 17, 221—239 (1945).

of the classical ergodic theorem. This may be the cause that RAIKOFF's special case $\lambda_n = a^n$ is easier to deal with than KHINTCHINE's case $\lambda_n = n$.

KAC, SALEM and ZYGMUND [4]) proved a generalisation of RAIKOFF's result, considering lacunary sequences (2), where λ_n need not be an integer, but where

$$\lambda_{n+1} \geqq q\,\lambda_n\,,$$

q denoting an arbitrarily given constant > 1. They proved that (3) holds almost everywhere in x, if $g(t)$ denotes a function of the class L^2, the Fourier coefficients of which satisfy a certain condition.

As far as I know, further results concerning the above problem are unknown, [5]) even in the case $\lambda_n = n$.

1. 2. In this paper I shall consider a general class of sequences $(f(n, x))$ $(n = 1, 2, \ldots;\ 0 \leqq x \leqq 1)$ and a general class of periodic functions $g(x) \in L^2$ and I shall prove that the relation

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(f(n, x)) = \int_0^1 g(t)\,dt \tag{4}$$

holds almost everywhere in x $(0 \leqq x \leqq 1)$ (Theorem 2).

The class of sequences $(f(n, x))$ includes e.g.

a. the case $f(n, x) = xn$
b. the case $f(n, x) = \lambda_n x$
if (λ_n) is any sequence (2) of real numbers satisfying
$\lambda_{n+1} - \lambda_n \geqq \delta > 0$, where δ is a constant,
c. the case $f(n, x) = (1 + x)^n$.

Obviously all these cases are included in the following theorem 1, which itself, as we shall prove in 1. 4, is a special case of the main theorem 2.

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[4]) M. KAC, R. SALEM and A. ZYGMUND, A gap theorem. Trans. Amer. Math. Soc. **63**, 235—243 (1948).

[5]) After the finishing of the manuscript, I discovered the interesting recent paper of P. ERDÖS, On the strong law of large numbers. Trans Amer. Math. Soc. **67** 51—56 (1950).

Mr ERDÖS deals with the special lacunary case (2) which also has been treatea by KAC-SALEM-ZYGMUND (l.c. [4])) and gives an interesting improvement of their result. But still more interesting is his Theorem 1, which states that there exists a function $g(x) \in L^2$, and a sequence of integers

$$\lambda_1 < \lambda_2 < \ldots$$

such that for almost all x

$$\frac{1}{N} \sum_{n=1}^{N} g(\lambda_n x) \to \infty\,.$$

It follows from ERDÖS's theorem that in my above Theorems 1, 2 in any case some condition on the Fourier coefficients, like (13), is necessary.

Theorem 1. *Let δ denote a positive constant. Let $f(1, x)$, $f(2, x), \dots$ be a sequence of real numbers defined for every value of x $(0 \leqq x \leqq 1)$, such that, for each $n = 1, 2, \dots$, the function $f(n, x)$ of x has a continuous derivative $f'_x \geqq \delta$ which is either a non-increasing or a non-decreasing function of x, whereas the expression $f'_x(n_1, x) - f'_x(n_2, x)$ for each couple of positive integers $n_1 \neq n_2$ is either a non-increasing or a non-decreasing function of x for $0 \leqq x \leqq 1$, the absolute value of which is $\geqq \delta$.*

Let $g(x) \in L^2$ denote a periodic function of period 1, and putting

$$g(x) \approx c_0 + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} c_k e^{2\pi i k x} \qquad (c_{-k} = \bar{c}_k),$$

let

$$\sum_{n=N+1}^{\infty} |c_n|^2 = O(\eta(N)),$$

where $\eta(N)$ denotes a positive non-increasing function, such that $\sum \frac{\eta(N}{N}$ converges. Then (4) holds almost everywhere in $0 \leqq x \leqq 1$.

1. 3. We now state the main result of the paper.

Definition 1. *Let $f(n, x)$ for $n = 1, 2, \dots$, be a real differentiable function of x for $0 \leqq x \leqq 1$, the derivative of which is either a non-increasing or a non-decreasing, positive function of x. Then put*

$$\Lambda_n = \operatorname{Max}\left(\frac{1}{f'(n, 0)}, \frac{1}{f'(n, 1)}\right). \tag{5}$$

Further, let for each couple of integers $n_1 \geqq 1$, $n_2 \geqq 1$, $n_1 \neq n_2$ the derivate Φ'_x of the function

$$\Phi = \Phi(n_1, n_2, x) = f(n_1, x) - f(n_2, x) \quad (n_1 \neq n_2) \tag{6}$$

be $\neq 0$ and either a non-increasing, or a non-decreasing function of x in $0 \leqq x \leqq 1$.

Then put

$$A(M, N) = \frac{1}{N^2} \sum_{n_1=M+2}^{M+N} \sum_{n_2=M+1}^{n_1-1} \operatorname{Max}\left(\frac{1}{\Phi'_x(n_1, n_2, 0)}, \frac{1}{\Phi'_x(n_1, n_2, 1)}\right) \tag{7}$$

for all integers $M \geqq 0$ and $N \geqq 1$.

Definition 2. *Let $g(x)$ denote a periodic function of the class L^2 in $0 \leqq x \leqq 1$ with period 1 and mean value 0, i.e.* [6])

$$g(x) \approx \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x} \qquad (c_0 = 0;\ c_{-k} = \bar{c}_k\ (k \geqq 1)), \tag{8}$$

and put

$$R_m = \sum_{k=m+1}^{\infty} |c_k|^2 \qquad (m \geqq 0). \tag{9}$$

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[6]) As the assumption $c_0 = \int_0^1 g(x)\, dx = 0$ may be made without loss of generality, we shall put $c_0 = 0$ for the rest of the paper.

Theorem 2.

1. *Let* C_1, C_2, C_3 *denote suitably chosen absolute constants. Let* $\eta_1(n)$ *and* $\eta_2(n)$ *denote two positive non-increasing functions of* $n = 1, 2, \ldots$ *such that*

$$(10) \qquad \sum_{n=1}^{\infty} \frac{\eta_1(n)}{n} < \infty; \quad \sum_{n=1}^{\infty} \frac{\eta_2(n)}{n} < \infty.$$

2. *Let* $f(n, x)$ $(n = 1, 2, \ldots)$ *denote the functions of Definition* 1 *and let*

$$(11) \qquad \sum_{n=M+1}^{M+N} \Lambda_n \leqq C_1 N \qquad (N \geqq 1)$$

$$(12) \qquad A(M, N) \leqq C_2 \eta_1^2(N) \qquad (M \geqq 0,\ N \geqq 1).$$

3. *Let* $g(x)$ *denote the function of Definition* 2 *and let*

$$(13) \qquad R_m \leqq C_3 \eta_2(m) \qquad (m \geqq 1).$$

Then

$$(14) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(f(n, x)) = 0.$$

1. 4. It is clear that the functions $f(n, x)$ of theorem 1 satisfy the conditions of theorem 2, for if one ranges the N numbers

$$f'_x(M+1, x),\ f'_x(M+2, x), \ldots,\ f'_x(M+N, x)$$

in increasing order, the difference between each two consecutive ones is at least δ, and we find

$$\sum_{n=M+1}^{M+N} \frac{1}{f'_x(n, x)} \leqq \sum_{\mu=1}^{N} \frac{1}{\mu\delta} < \frac{1}{\delta} \log 3N.$$

$$\sum_{n_2=M+1}^{n_1-1} \frac{1}{|f'_x(n_1, x) - f'_x(n_2, x)|} \leqq 2 \sum_{\mu=1}^{N} \frac{1}{\mu\delta} < \frac{2}{\delta} \log 3N.$$

Hence

$$\sum_{n=M+1}^{M+N} \Lambda_n \leqq \sum_{n=M+1}^{M+N} \frac{1}{f'_x(n, 0)} + \sum_{n=M+1}^{M+N} \frac{1}{f'_x(n, 1)} \leqq \frac{2}{\delta} \log 3N$$

and

$$A(M, N) \leqq \frac{4}{\delta} \frac{\log 3N}{N}.$$

1. 5. With a different method, which only holds for the case $f(n, x) = \lambda_n x$, where the λ_n are integers satisfying (2), I have proved a theorem, which is somewhat sharper than Theorem 1, as the inequality for $\sum_{n=N+1}^{\infty} |c_n|^2$ is replaced by a weaker one. I shall publish those results elsewhere.

§ 2. *Some Lemma's.*

2. 1. Lemma 1. *Let the function*

(15) $$g(x) \approx \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x} \qquad (c_0 = 0,\ c_{-k} = c_k;\ k \geqq 1)$$

belong to the class L^2 *and put*

(16) $$R_m = \sum_{k=m+1}^{\infty} |c_k|^2 \qquad (m \geqq 0)$$

and

(17) $$G(x, a) = \frac{1}{2a} \int_{-a}^{a} g(x+t)\, dt \qquad (a > 0).$$

Then

(18) $$G(x, a) \approx \sum_{k=-\infty}^{\infty} C_k e^{2\pi i k x},$$

where

(19) $$C_k = \frac{\sin 2\pi k a}{2\pi k a} c_k \qquad (k \lesseqgtr 0),$$

(20) $$\sum_{k=1}^{\infty} |C_k|^2 \leqq R_0,$$

(21) $$\sum_{k=p+1}^{\infty} |C_k| \leqq \frac{1}{2\pi a} \sqrt{\frac{R_p}{p}} \quad (p \geqq 1), \qquad \sum_{k=1}^{\infty} |C_k| \leqq \frac{\sqrt{R_0}}{4a},$$

whereas the relation

(22) $$G(x, a) = \sum_{k=-\infty}^{\infty} C_k e^{2\pi i k x}$$

holds uniformly in x.

Remark. The condition $g(x) \in L^2$ is not required for the proof of all the statements of the lemma, but a refinement in that direction would be of no use for our purpose.

Proof. As is proved in textbooks on Fourier-series, [7]) any Fourier-series, whether convergent or not, may be integrated term by term between any limits; i.e. the sum of the integrals of the separate terms, is the integral of the function of which the series is the Fourier series. Now applying this process to the function

$$g(x+t) \approx \sum_{k=-\infty}^{\infty} (c_k e^{2\pi i k x}) \cdot e^{2\pi i k t},$$

integrating with respect to t between the limits $-a$ and a, we immediately find (22) and (19). We now shall prove the formula (21), from which it is immediately clear that (22) holds uniformly in x.

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[7]) E.g. cf. E. C. Titchmarsh, The theory of functions, Oxford Univ. Press, (1939).

From (19) we have for $p \geqq 0$, $q \geqq 1$

$$\sum_{k=p+1}^{p+q} |C_k| \leqq \frac{1}{2\pi a} \sum_{k=p+1}^{p+q} |c_k| \cdot \frac{1}{k} \leqq \frac{1}{2\pi a} \Big(\sum_{k=p+1}^{p+q} |c_k|^2 \Big)^{1/2} \Big(\sum_{k=p+1}^{p+q} \frac{1}{k^2} \Big)^{1/2},$$

using the CAUCHY–SCHWARZ–RIEMANN-inequality; hence

$$\sum_{k=1}^{\infty} |C_k| \leqq \frac{1}{2\pi a} \frac{\pi}{\sqrt{6}} \sqrt{R_0} < \frac{\sqrt{R_0}}{4a}$$

and, if $p \geqq 1$:

$$\sum_{k=p+1}^{p+q} |C_k| \leqq \frac{1}{2\pi a} R_p^{1/2} \Big(\int_p^{\infty} \frac{du}{u^2} \Big)^{1/2} = \frac{1}{2\pi a} \sqrt{\frac{R_p}{p}},$$

which proves (21).

Now from the fact that (22) holds uniformly in x we deduce immediately that the right hand side is the Fourier series of its sum, which proves (18). Finally we note that (20) is an immediate consequence of (16) and (19) because of

$$\left| \frac{\sin 2\pi ka}{2\pi ka} \right| \leqq 1.$$

Lemma 2. *If $g(x) \in L^2$ and $G(x, a)$ $(a > 0)$ denote the functions of* (15) *and* (17) *and if R_m is defined by* (16) $(m \geqq 0)$, *we have*

$$\int_0^1 |g(x) - G(x, a)|^2 \, dx \leqq 100\, R_0\, m^4 a^4 + 8 R_m, \tag{23}$$

for each integer $m \geqq 1$ which satisfies

$$2\pi m a \leqq 1.$$

Proof. By Lemma 1 we have (see (15) and (18))

$$g(x) - G(x, a) \approx \sum_{k=-\infty}^{\infty} (c_k - C_k)\, e^{2\pi i k x},$$

hence

$$g(x) - G(x, a) \approx \sum_{k=-\infty}^{\infty} c_k \Big(1 - \frac{\sin 2\pi ka}{2\pi ka}\Big) e^{2\pi i k x},$$

by (19). Now, since $g(x)$ and $G(x, a)$ belong to L^2 ($G(x, a)$ being a continuous function of x), we have, by PARSEVAL's theorem,

$$\begin{cases} \displaystyle\int_0^1 |g(x) - G(x, a)|^2 \, dx = 2 \sum_{k=1}^{\infty} |c_k|^2 \Big(1 - \frac{\sin 2\pi ka}{2\pi ka}\Big)^2 = \\ \displaystyle = 2 \sum_{k=1}^{m} |c_k|^2 \Big(1 - \frac{\sin 2\pi ka}{2\pi ka}\Big)^2 + 2 \sum_{k=m+1}^{\infty} |c_k|^2 \Big(1 - \frac{\sin 2\pi ka}{2\pi ka}\Big)^2. \end{cases} \tag{24}$$

Now we note that

$$\Big(1 - \frac{\sin 2\pi ka}{2\pi ka}\Big)^2 \leqq 4$$

and also that

$$\left(1-\frac{\sin 2\pi ka}{2\pi ka}\right)^2 \leqq \frac{(2\pi ka)^4}{(3!)^2} < \frac{(6,5)^4 m^4 a^4}{6^2} < 100\, m^4 a^4,$$

if $1 \leqq k \leqq m$, and $2\pi m a \leqq 1$.

Therefore we have by (24)

$$\int_0^1 |g(x) - G(x, a)|^2 dx \leqq 100\, m^4 a^4 \sum_{k=1}^{m} |c_k|^2 + 8 \sum_{k=m+1}^{\infty} |c_k|^2 \leqq$$

$$\leqq 100\, m^4 a^4 R_0 + 8\, R_m.$$

Q. e. d.

Lemma 3. *Let $f(n, x)$ for $n = 1, 2, \ldots$ denote a real continuous function of x for $a \leqq x \leqq b$, and let*

$$\Phi(n_1, n_2, x) = f(n_1, x) - f(n_2, x) \text{ for } n_1 \neq n_2$$

have a continuous derivative Φ'_x, which is $\neq 0$ and either non-decreasing or non-increasing for $a \leqq x \leqq b$.

Finally, put

$$A_N = \frac{1}{N^2} \sum_{n_1=2}^{N} \sum_{n_2=1}^{n_1-1} \operatorname{Max}\left(\frac{1}{\Phi'_x(n_1, n_2, a)}, \frac{1}{\Phi'_x(n_1, n_2, b)}\right).$$

Then we have for $N \geqq 2$, $h > 0$ (h not depending on n and x)

$$\int_a^b \Big|\sum_{n=1}^{N} e^{2\pi i h f(n,x)}\Big|^2 dx \leqq (b-a) N + \frac{A_N}{h} N^2.$$

This lemma I have proved in a previous paper. [8]) I deduce from it the following

Lemma 4. *Let $f(n, x)$ for $n = 1, 2, \ldots$ denote the functions of Definition 1, and let $A(M, N)$ be defined by (7). Then for each integer $h \neq 0$ which does not depend on n and x we have*

$$\int_0^1 \Big|\sum_{n=M+1}^{M+N} e^{2\pi i h f(n,x)}\Big|^2 dx \leqq N + \frac{A(M, N)}{h} N^2. \qquad (M \geqq 0,\ N \geqq 1).$$

Proof. For $M \geqq 0$ consider the sequence

$$f(M+1, x),\ f(M+2, x), \ldots$$

These functions satisfy the conditions of Lemma 3 with $f(M+n, x)$ in stead of $f(n, x)$ and with $a = 0$, $b = 1$. The corresponding number A_N defined in Lemma 3 ($a = 0$, $b = 1$) is identical with the number $A(M, N)$ which was defined by (7). Therefore Lemma 4 is an immediate consequence of Lemma 3.

Lemma 5. *Let $f(n, x)$ for $n = 1, 2, \ldots$ denote the functions of*

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[8]) J. F. Koksma, Ein mengentheoretischer Satz über die Gleichverteilung modulo Eins. Comp. Math. 2, 250—258 (1935).

Definition 1, *let* $g(x)$ *denote the function of Definition* 2, *let* $G(x, a)$ $(a > 0)$ *denote the function which is defined by* (17) *and finally, let*

$$S(M, N, x; a) = \sum_{n=M+1}^{M+N} G(f(n, x), a). \tag{25}$$

Then for each integer $p \geqq 1$

$$\left\{\begin{aligned} &\int_0^1 |S(M, N, x; a|^2\, dx \leqq 4 R_0 Np + 8 R_0 N^2 A(M, N) \log 3p + \\ &+ \frac{1}{\pi a^2} N^{3/2} \sqrt{\frac{R_0 R_p}{p}} + \frac{8}{\pi a} N^2 \sqrt{\frac{R_0 R_p}{p} A(M, N) \log 3p} + \frac{1}{\pi^2 a^2} N^2 \frac{R_p}{p}, \end{aligned}\right. \tag{26}$$

where $A(M, N)$ *and* R_p *are defined by* (7) *and* (9) (*Definition* 1 *and* 2).

Proof. Using Lemma 1, we deduce from (22) and (25)

$$S(M, N, x; a) = \sum_{n=M+1}^{M+N} \sum_{k=-\infty}^{\infty} C_k e^{2\pi i k f(n,x)} = \sum_{k=-\infty}^{\infty} C_k \sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)}.$$

Hence

$$|S(M, N, x; a)| \leqq 2 \sum_{k=1}^{\infty} |C_k| \,|\sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)}| \leqq$$

$$\leqq 2 \sum_{k=1}^{p} |C_k| \,|\sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)}| + 2N \sum_{k=p+1}^{\infty} |C_k|$$

$(p \geqq 1)$, as the inner sum is in absolute value $\leqq N$.

Therefore we have

$$|S(M, N, x; a|^2 \leqq 4 \Big(\sum_{k=1}^{p} |C_k| \,|\sum_{n=M=1}^{M+N} e^{2\pi i k f(n,x)}| \Big)^2 +$$

$$+ 8N \Big(\sum_{k=p+1}^{\infty} |C_k| \Big) \Big(\sum_{k=1}^{p} |C_k| \,|\sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)}| \Big) + 4N^2 \Big(\sum_{k=p+1}^{\infty} |C_k| \Big)^2$$

and applying the Cauchy–Schwarz–Riemann-inequality

$$|S(M, N, x; a)|^2 \leqq 4 \Big(\sum_{k=1}^{p} |C_k|^2 \Big) \Big(\sum_{k=1}^{p} |\sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)}|^2 \Big) +$$

$$+ 8N \Big(\sum_{k=p+1}^{\infty} |C_k| \Big) \Big(\sum_{k=1}^{p} |C_k| \,|\sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)}| \Big) + 4N^2 \Big(\sum_{k=p+1}^{\infty} |C_k| \Big)^2.$$

Integrating this and applying (20) and (21) we find:

$$\left\{\begin{aligned} &\int_0^1 |S(M, N, x; a)|^2\, dx \leqq 4 R_0 \sum_{k=1}^{p} \int_0^1 |\sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)}|^2\, dx + \\ &+ \frac{4N}{\pi a} \sqrt{\frac{R_p}{p}} \sum_{k=1}^{p} |C_k| \int_0^1 |\sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)}|\, dx + \frac{N^2}{\pi^2 a^2} \cdot \frac{R_p}{p}. \end{aligned}\right. \tag{27}$$

Now we have, by Lemma 4,

$$\int_0^1 |\sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)}|^2\, dx \leqq N + \frac{A(M, N)}{k} N^2.$$

Moreover applying the CAUCHY–SCHWARZ–RIEMANN-inequality for integrals, we find

$$\int_0^1 \Big| \sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)} \Big| \, dx \leqq \Big(\int_0^1 1^2 \, dx\Big)^{1/2} \cdot \Big(\int_0^1 \Big| \sum_{n=M+1}^{M+N} e^{2\pi i k f(n,x)} \Big|^2 dx\Big)^{1/2}$$

$$\leqq \sqrt{N + \frac{A(M,N)}{k} N^2} \leqq \sqrt{N} + N \sqrt{\frac{A(M,N)}{k}},$$

by MINKOWSKI's inequality. Therefore it follows from (27)

$$\int_0^1 |S(M,N,x;\alpha)|^2 \, dx \leqq 4 R_0 \sum_{k=1}^{p} \Big(N + \frac{A(M,N)}{k} N^2\Big) +$$

$$+ \frac{4N}{\pi \alpha} \sqrt{\frac{R_p}{p}} \sum_{k=1}^{p} |C_k| \Big(\sqrt{N} + N \sqrt{\frac{A(M,N)}{k}}\Big) + \frac{N^2}{\pi^2 \alpha^2} \cdot \frac{R_p}{p}$$

and hence, by (21), since

$$\sum_{k=1}^{p} |C_k| \cdot \frac{1}{\sqrt{k}} \leqq \Big(\sum_{k=1}^{p} |C_k|^2\Big)^{1/2} \Big(\sum_{k=1}^{p} \frac{1}{k}\Big)^{1/2} < 2 \sqrt{R_0} \sqrt{\log 3p},$$

we have

$$\int_0^1 |S(M,N,x;\alpha)|^2 \, dx \leqq 4 R_0 N p + 8 R_0 N^2 A(M,N) \log 3p +$$

$$+ \frac{N^{3/2} \sqrt{R_0}}{\pi \alpha^2} \sqrt{\frac{R_p}{p}} + \frac{8 N^2}{\pi \alpha} \sqrt{R_0} \sqrt{\frac{R_p}{p}} \sqrt{A(M,N)} \sqrt{\log 3p} + \frac{N^2}{\pi^2 \alpha^2} \frac{R_p}{p},$$

which proves (26).

Lemma 6. *Let $\Omega(x) \geqq 0$ denote a periodic function with period* 1, *which is Lebesgue-integrable. Let $\psi(x)$ denote a differentiable function in* $(0, 1)$, *such that $\psi'(x)$ is a positive and either a non-increasing, or a non-decreasing function of x in* $(0, 1)$. *Put*

$$\Lambda = \operatorname{Max}\Big(\frac{1}{\psi'(0)}, \frac{1}{\psi'(1)}, 1\Big). \tag{28}$$

Then we have

$$\int_0^1 \Omega(\psi(x)) \, dx \leqq 8 \Lambda \int_0^1 \Omega(u) \, du. \tag{29}$$

Proof. If $\Psi(u)$ denotes the inverse function of $u = \psi(x)$, we have

$$\int_0^1 \Omega(\psi(x)) \, dx = \int_{\psi(0)}^{\psi(1)} \Omega(u) \, \Psi'(u) \, du. \tag{30}$$

We now distinguish two cases.

A. If

$$\psi(1) - \psi(0) \leqq 8,$$

then the right hand side of (30) is

$$\leqq \Lambda \int_{\psi(0)}^{\psi(1)} \Omega(u) \, du \leqq 8 \Lambda \int_0^1 \Omega(u) \, du,$$

because of (28) and the fact that $\Omega \geqq 0$ is periodic. In this case therefore (29) has been proved already.

B. If

$$\psi(1) - \psi(0) > 8,$$

then we write (30) in the form

$$(31) \qquad \begin{cases} \int\limits_0^1 \Omega(\psi(x))\, dx = \sum\limits_{\nu=A}^{B} \int\limits_{\nu}^{\nu+1} \Omega(u)\, \Psi'(u)\, du + \\ + \int\limits_{\psi(0)}^{A} \Omega(u)\, \Psi'(u)\, du + \int\limits_{B}^{\psi(1)} \Omega(u)\, \Psi'(u)\, du, \end{cases}$$

where we have put

$$A = [\psi(0)] + 2 \quad , \quad B = [\psi(1)] - 3.$$

Now by the same argument as we have used in *A*, we have

$$(32) \qquad \int\limits_{\psi(0)}^{A} \Omega(u)\, \Psi'(u)\, du \leqq 3\,\Lambda \int\limits_0^1 \Omega(u)\, du,$$

$$(33) \qquad \int\limits_{B}^{\psi(1)} \Omega(u)\, \Psi'(u)\, dn \leqq 4\,\Lambda \int\limits_0^1 \Omega(u)\, du.$$

We now deal with the sum on the right hand side of (31) and write

$$\sum_{\nu=A}^{B} \int\limits_{\nu}^{\nu+1} \Omega(u)\, \Psi'(u)\, du = \sum_{\nu=A}^{B} \int\limits_0^1 \Omega(v+\nu)\, \Psi'(v+\nu)\, dv =$$

$$= \int\limits_0^1 \Omega(v) \{ \sum_{\nu=A}^{B} \Psi'(v+\nu) \}\, dv \leqq \{ \sum_{\nu=A}^{B+1} \Psi'(\nu) \} \int\limits_0^1 \Omega(v)\, dv,$$

since $\Psi'(u)$ is either non-increasing of non-decreasing. Now

$$\sum_{\nu=A}^{B+1} \Psi'(\nu) \leqq \int\limits_{A-1}^{B+2} \Psi'(u)\, du < \int\limits_{\psi(0)}^{\psi(1)} \Psi'(u)\, du = [\Psi(u)]_{\psi(0)}^{\psi(1)} = 1,$$

and therefore

$$(34) \qquad \sum_{\nu=A}^{B} \int\limits_{\nu}^{\nu+1} \Omega(u)\, \Psi'(u)\, du \leqq \int\limits_0^1 \Omega(u)\, du.$$

Combining (31), (32), (33), (34), we find

$$\int\limits_0^1 \Omega(\psi(x))\, dx \leqq 8\,\Lambda \int\limits_0^1 \Omega(u)\, du,$$

because of $\Lambda \geqq 1$. Q. e. d.

Lemma 7. *If $f(n, x)$ $(n = 1, 2, \ldots)$ and $g(x)$ denote the functions of Definition* 1 *and Definition* 2, *and if we put*

$$(35) \qquad S^*(M, N, x) = \sum_{n=M+1}^{M+N} g(f(n, x)),$$

where $M \geqq 0$, $N \geqq 1$ are integers, we have for every positive a, and every positive integer m, satisfying $2\pi m a \leqq 1$ and for every positive integer p the inequality

$$(36)\quad \begin{cases} \{\int\limits_0^1 |S^*(M,N,x)|^2 dx\}^{1/2} \leqq \{(100 R_0 m^4 a^4 + 8 R_m) N \sum\limits_{n=M+1}^{M+N} 8(\Lambda_n + 1)\}^{1/2} + \\ + \{4 R_0 N p + 8 R_0 N^2 A(M,N) \log 3p + \\ \qquad + \frac{N^{3/2}}{\pi a^2}\sqrt{\frac{R_0 R_p}{p}} + \frac{8N^2}{\pi a}\sqrt{\frac{R_0 R_p}{p} A(M,N)\log 3p} + \frac{N^2}{\pi^2 a^2}\frac{R_p}{p}\Big\}^{1/2}, \end{cases}$$

using the notations of Definitions 1, 2.

Proof. If $G(x, a)$ $(a > 0)$ is defined by (17) and if $S(M, N, x, a)$ denotes the sum (25), we find from (35) by MINKOWSKI's inequality

$$(37)\quad \begin{cases} \{\int\limits_0^1 |S^*(M,N,x)|^2 dx\}^{1/2} \leqq \{\int\limits_0^1 |S^*(M,N,x) - S(M,N,x;a)|^2 dx\}^{1/2} + \\ \qquad + \{\int\limits_0^1 |S(M,N,x;a)|^2 dx\}^{1/2}. \end{cases}$$

Now by (25) and (35)

$$(38)\quad \begin{cases} \int\limits_0^1 |S^*(M,N,x) - S(M,N,x;a)|^2 dx = \\ = \int\limits_0^1 |\sum\limits_{n=M+1}^{M+N} (g(f(n,x)) - G(f(n,x),a))|^2 dx \leqq \\ \qquad \leqq N \sum\limits_{n=M+N}^{M+N} \int\limits_0^1 |g(f(n,x)) - G(f(n,x),a)|^2 dx \end{cases}$$

using the CAUCHY–SCHWARZ–RIEMANN-inequality for sums. Applying Lemma 6 with

$$\Omega(x) = |g(x) - G(x, a)|^2, \quad \psi(x) = f(n, x),$$

we find

$$\int\limits_0^1 |g(f(n,x)) - G(f(n,x),a)|^2 dx \leqq 8(\Lambda_n + 1)\int\limits_0^1 |g(u) - G(u,a)|^2 du,$$

where Λ_n is defined by (5). Therefore we have by Lemma 2

$$\int\limits_0^1 |g(f(n,x)) - G(f(n,x),a)|^2 dx \leqq 8(\Lambda_n + 1)(100 R_0 m^4 a^4 + 8 R_m)$$

tor every integer $m \geqq 1$ which satisfies $2\pi m a \leqq 1$. Therefore we have by (38)

$$\int\limits_0^1 |S^*(M,N,x) - S(M,N,x;a)|^2 dx \leqq (100 R_0 m^4 a^4 + 8 R_m) N \sum\limits_{n=M+1}^{M+N} 8(\Lambda_n + 1)$$

and thus we find from (37)

$$(39)\quad \begin{cases} \{\int\limits_0^1 |S^*(M,N,x)|^2 dx\}^{1/2} \leqq \{(100 R_0 m^4 a^4 + 8 R_m) N \sum\limits_{n=M+1}^{M+N} 8(\Lambda_n + 1)\}^{1/2} + \\ \qquad + \{\int\limits_0^1 |S(M,N,x;a)|^2 dx\}^{1/2}. \end{cases}$$

Now applying Lemma 5 ($p \geqq 1$), we find from (39) and (26) immediately (36), which proves Lemma 7.

Lemma 8. *Be $\eta(x) > 0$ and non-increasing for $x \geqq 1$, such that*

$$\sum_{N=1}^{\infty} \frac{\eta(N)}{N} < \infty. \tag{40}$$

Then we have for any positive ε

$$\sum_{N=1}^{\infty} \frac{\eta(N^{\varepsilon})}{N} < \infty \tag{41}$$

and also

$$\eta(N) \log N \to 0\,, \text{ as } N \to \infty. \tag{42}$$

Proof. *A.* It is clear that $\frac{\eta(N^{\varepsilon})}{N}$ is a non-increasing function of N and therefore

$$\sum_{N=1}^{\infty} \frac{\eta(N^{\varepsilon})}{N} < \infty\,, \text{ if } \int_1^{\infty} \frac{\eta(t^{\varepsilon})}{t}\, dt \text{ converges.}$$

Now putting $t^{\varepsilon} = u$, we find $\frac{dt}{t} = \frac{1}{\varepsilon}\frac{du}{u}$, hence

$$\int_1^{T} \frac{\eta(t^{\varepsilon})}{t}\, dt = \frac{1}{\varepsilon} \int_1^{T^{\varepsilon}} \frac{\eta(u)}{u}\, du\,,$$

which proves the first assertion, since $\sum_{N=1}^{\infty} \frac{\eta(N)}{N} < \infty$.

B. As $\eta(N)/N$ is non-increasing and as $\sum_{N=1}^{\infty} \frac{\eta(N)}{N} < \infty$, we conclude from a well known theorem that also the series

$$\sum_{n=1}^{\infty} \eta(2^n) < \infty.$$

Since the general term of this series is a non-increasing function of n, we have by another well known theorem

$$n\, \eta(2^n) \to 0\,, \text{ as } n \to \infty.$$

Now let N denote a positive integer > 2 and be n the integer for which

$$2^n \leqq N < 2^{n+1}.$$

Then we have

$$\eta(N) \log N \leqq \eta(2^n)(n+1) \log 2 \to 0 \text{ as } N \to \infty.$$

Q. e. d.

Lemma 9. *Let $\eta_1(N)$ and $\eta_2(N)$ be positive non-increasing functions such that*

$$\sum_{N=1}^{\infty} \frac{\eta_1(N)}{N} < \infty, \quad \sum_{N=1}^{\infty} \frac{\eta_2(N)}{N} < \infty.$$

Then $\sqrt{\eta_1(N)\,\eta_2(N)}$ is also a non-increasing function of N, such that

$$\sum_{N=1}^{\infty} \frac{\sqrt{\eta_1(N)\,\eta_2(N)}}{N} < \infty.$$

Proof. It is trivial that $\sqrt{\eta_1(N)\,\eta_2(N)}$ is a non-increasing function. Further by the CAUCHY–SCHWARZ–RIEMANN-inequality

$$\sum_{N=1}^{N*} \frac{\sqrt{\eta_1(N)\,\eta_2(N)}}{N} \leqq \sqrt{\sum_{N=1}^{N*} \frac{\eta_1(N)}{N}} \cdot \sqrt{\sum_{N=1}^{N*} \frac{\eta_2(N)}{N}},$$

which proves the lemma.

Finally we use a lemma, which is a special case of a theorem due to I. S. GÁL and the author [9]), and which has also been proved in a joint paper by R. SALEM and the author: [10])

Lemma 10. Let $f_n(x) \in L^p(0, 1)$ $(n = 1, 2, \ldots)$ $(p > 1)$ *be a sequence of functions such that*

$$\int_0^1 \Big| \sum_{n=M+1}^{M+N} f_n(x) \Big|^p dx < C(M+N)^{p-\sigma} N^{\sigma} \eta(N),$$

where $C > 0$, $\sigma > 1$ are constants and where $\eta(N)$ denotes a positive non-increasing function of N such that $\sum \eta(N)/N < \infty$.

Then

$$\frac{1}{N} \sum_{n=1}^{N} f_n(x) \to 0, \text{ as } N \to (\infty),$$

almost everywhere in $(0, 1)$.

§ 3. *Proof of Theorem* 2.

We put $(N \geqq 1)$

$$a = \frac{1}{2\pi N^{1/4}}, \quad m = [N^{1/8}], \quad p = [N^{3/4}]. \tag{43}$$

[9]) I. S. GÁL and J. F. KOKSMA, Sur l'ordre de grandeur des fonctions sommables. These Proc. **53**, 638–653 (1950) = Indag. Math. **12**, Fasc. 3 (1950).

[10]) J. F. KOKSMA and R. SALEM, Uniform distribution and Lebesgue integration, Acta Sci. Math. Szeged. 12B, 87–96 (1950).

The main theorem of this joint paper is closely related with the above theorems, as it deals with the problem, whether from the uniform distribution modulo 1 of a sequence $u_1, u_2, \ldots,$ may be concluded that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(u_n + x) = \int_0^1 g(x)\,dx$$

for a given $g(x) \in L^2$ and for almost all x.

Then $m \geqq 1$, $p \geqq 1$, $2\pi m a \leqq 1$.

Further we have by (13)

$$R_m = O(\eta_2(m)) = O(\eta_2(N^{1/8})), \tag{44}$$

$$R_p = O(\eta_2(p)) = O(\eta_2(N^{1/8})), \tag{45}$$

as $\eta_2(n)$ is non-increasing.

We now apply Lemma 7. Substituting (43), (44), (45), (11), (12) in (36) we find

$$\Bigl\{\int_0^1 |S^*(M, N, x)|^2\, dx\Bigr\}^{1/2} = O\bigl(\{N^{3/2} + N^2 \eta_2(N^{1/8})\}^{1/2}\bigr) +$$

$$+ O\bigl(\{N^{7/4} + N^2 \eta_1^2(N) \log 3N + N^{13/8} \sqrt{\eta_2(N^{1/8})} + N^{15/8} \sqrt{\eta_2(N^{1/8})\, \eta_1^2(N) \log 3N} + N^{7/4} \eta_2(N^{1/8})\}^{1/2}\bigr).$$

Hence using MINKOWSKI's inequality and applying Lemma 8, i.e.

$$\eta_1(N) \log 3N \to 0 \quad \text{and} \quad \eta_1(N) \to 0,\ \eta_2(N^{1/8}) \to 0,$$

we find

$$\Bigl\{\int_0^1 |S^*(M, N, x)|^2\, dx\Bigr\}^{1/2} = O\bigl(N^{15/16} + N\sqrt{\eta_2(N^{1/8})} + N\sqrt{\eta_1(N)}\bigr).$$

Hence by Lemma 9

$$\int_0^1 |S^*(M, N, x)|^2\, dx = O(N^2 \eta(N)),$$

where $\eta(N)$ is a positive non-increasing function such that

$$\sum_{N=1}^{\infty} \frac{\eta(N)}{N} < \infty.$$

Therefore by Lemma 10 and (35)

$$S^*(0, N, x) = o(N)$$

almost everywhere in $0 \leqq x \leqq 1$.

Q. e. d.