GEOGRAPHY, PHYSICAL

THEORY ON CENTRAL RECTILINEAR RECESSION OF SLOPES. III

BY

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Central rectilinear recession with decreasing h-values (crest-recession)

1. *Introduction.*

In the following lines, we shall subject the central rectilinear recession of crests with a straight-lined triangular profile FT_1H (fig. 12) [1]) to a mathematical treatment.

From the steep rockwall, in a very small unit of time, a part T_1FT_2, T_2PRT_3 etc. is removed, while in the same period a screes volume $I'PF$, $II'I'PR$ etc. is deposited on an almost horizontal form FII' at the foot. Supposing further the same conditions as in the first part of our theory (16, I, p. 961 — 962) [2]) the coordinates of the points P and R (fig. 12) are (x, y) and $(x + dx, y + dy)$. The constant base FD being k, the decreasing height

$$T_2D = k\,\frac{y}{x} \text{ and } T_3D = k\,\frac{y+dy}{x+dx}.$$

So

$$T_2T_3 = k\,\frac{y\,dx - x\,dy}{x\,(x+dx)}\,;\quad FT_3T_2 = \tfrac{1}{2}\,k^2\,\frac{y\,dx - x\,dy}{x\,(x+dx)}\,;$$

$$FSP = \frac{x^2}{k^2}\cdot FT_3T_2;\quad PST_3T_2 = \left(1-\frac{x^2}{k^2}\right) FT_3T_2 = \frac{k^2-x^2}{2\,x}\cdot\frac{y\,dx - x\,dy}{x+dx}.$$

Now, considering, that for

$$R \to P,\ \text{Lim.}\ \frac{II'\,I'\,PR}{II'\,I'\,PK} = 1 \text{ and Lim.}\ \frac{PRT_3T_2}{PST_3T_2} = 1$$

and putting

$$(1-c)\,II'\,I'\,PK = PST_3T_2$$

we have

$$y\,(1-c)\,(a\,dy - dx) = \frac{k^2-x^2}{2\,x^2}\,(y\,dx - x\,dy). \tag{17}$$

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[1]) See PHILIPPSON, A., literature (5, II, 2, p. 63) in the second part (p. 1162) and the introduction of the first part of our theory (p. 959—961).

[2]) The numbers in parentheses refer to the list of literature at the end of part IV of this article. For the literaturenumbers 1—15, see p. 1162 of part I—II of our theory.

It may be remembered, that in the case of central rectilinear recession of a slope, bordered at the top by a horizontal plateau with constant height h, we have found:

$$y\,(1-c)\,(a\,dy-dx)=\frac{h^2-y^2}{2\,y^2}\,(y\,dx-x\,dy). \tag{10}$$

(See 16, I, p. 962).

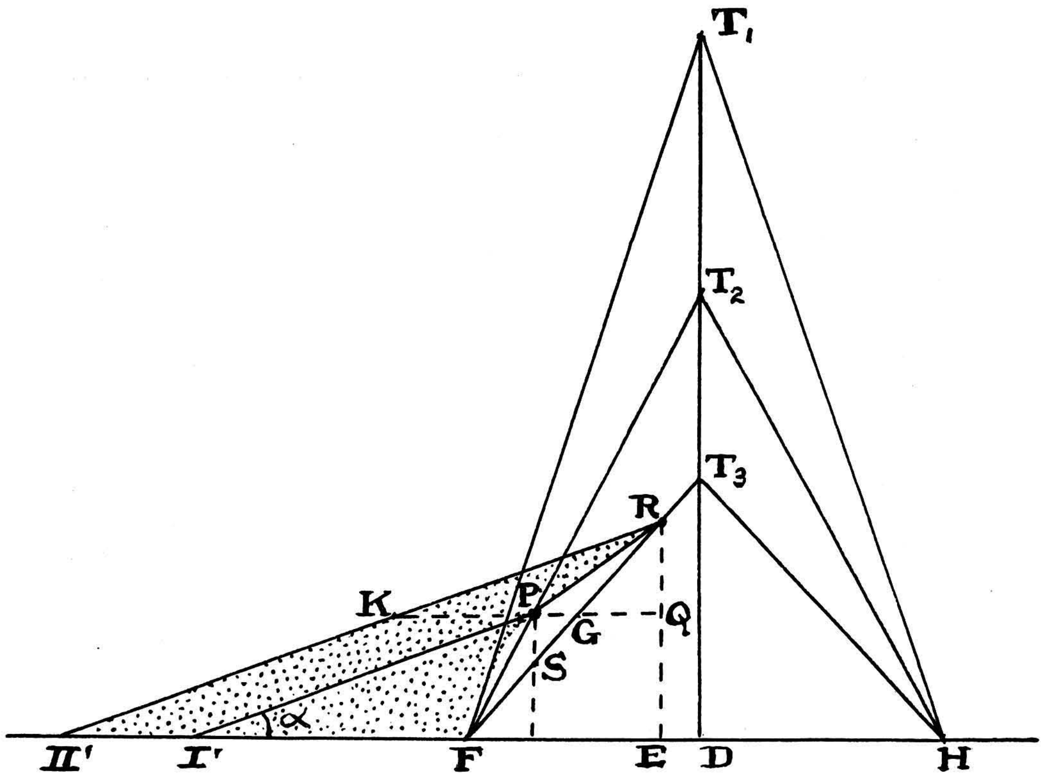

Fig. 12

2. *Projective-geometric treatment of our formulae* (10) *and* (17).

In our projective-geometric treatment of LEHMANN's theory on parallel recession of mountain slopes, we observed, that the form in which LEHMANN had cast his theory did not lend itself to a rapid construction of the curves required and a ready comparison with the phenomena observable in Nature (10, p. 541).

In our own theory on central rectilinear recession with stable position of the basis point of the slopes, it is mostly also fairly difficult and laborious to find and to draw the integral curves of our differential equations (10) and (17).

The geomorphologist wants a quick survey of possible rock-profiles underneath the screes, drawn if desired, with sufficient exactness for given values of β (initial slope-angle of the wall of a plateau or crest), a (slope-angle of the screes) and c (ratio between rock-volume and screes-volume; 16, I, p. 961 — 962). Therefore, we will join the well-known graphical solution of a differential equation of the first degree by means of isoclines (17, p. 124 — 126) to a nomographical construction of these isoclines. In this way we find a method for discussing the possible forms of the rock-profile underneath the screes for given values of a ($= \cot\alpha$), b ($= \cot\beta$) and c, *without drawing any curve* (cf. part 6: The ratio between the flat part ...).

Although several integral curves of the equation

$$y\,(1-c)\,(a\,dy-dx) = \frac{h^2-y^2}{2\,y^2}\,(y\,dx-x\,dy) \tag{10}$$

have already been discussed in the second part of our theory (16, II, p. 1154 — 1161) we shall begin to treat it again with the new method, in order to have a control. The more so as the general form of both equations (10) and (17) is the same.

By introducing $\frac{dy}{dx} = p$, the equation (10) may be written

$$y\,(1-c)\,(ap-1) = \frac{h^2-y^2}{2\,y^2}\,(y-px)$$

or

$$y^3\{2(1-c)\;(ap-1)+1\} - h^2y = -\,px\;(h^2-y^2).$$

Putting $2(1-c)\;(ap-1)+1 = m$, we have

$$px = y\,\frac{h^2-my^2}{h^2-y^2} \tag{18}$$

and in the same way for central linear crest-weathering

$$px = y\,\frac{k^2-mx^2}{k^2-x^2}. \tag{19}$$

For the constant p-value, the curve of formula (18) is the locus of points with line-elements of the integral-curves of the differential-equation

$$\frac{dy}{dx} = \frac{y}{x}\cdot\frac{h^2-my^2}{h^2-y^2},$$

parallel to p (isocline).

The construction of the desired rock-profile may be obtained by drawing a family of such isoclines (formula 18) for p-values from $\tan\beta$ to $\tan\alpha$, decreasing by $d = 0{,}1$ or a smaller interval and joining the successive elements, $p = \tan\beta$ being in the footpoint of the slope we started from (= zero point of our coordination system). Each new element $p = 0{,}1\,n$ has to be drawn in the point of intersection of the element $p = 0{,}1\,(n+1)$ with the isocline $p = 0{,}1\,n$ (cf.: sub 4, construction of the integral-curve).

The same may be said for the construction of the integral-curve of

$$\frac{dy}{dx} = \frac{y}{x}\cdot\frac{k^2-mx^2}{k^2-x^2}$$

(considering $\frac{dy}{dx} = \tan\beta$ for $x = 0$ and $y = 0$) by means of the family of isoclines

$$px = y\,\frac{k^2-mx^2}{k^2-x^2}.$$

3. *Construction of the isoclines* (*curve of formula* 18 *for* $p =$ *constant*).
The curve of formula (18), written in the form

$$y^2 (my - px) + h^2 (px - y) = 0$$

has a tangent $y = px$ in the zeropoint of our coordination-system and the asymptotes $y = h$ and $y = \frac{p}{m} x$.

Putting $s = \frac{h^2 - my^2}{h^2 - y^2}$ we get (from form. 18)

$$\frac{p}{s} = \frac{y}{x}$$

p being given and m depending upon given values of a, c and p, s may be obtained for a chosen value y_0 of y by projective transformation of a quadratic scale of y. The direction of $\frac{y_0}{x_0}$ thus being given, x_0 is known.

Changing y into x, the same may be said for the drawing of the curve $px = y \; \frac{k^2 - mx^2}{k^2 - x^2}$.

In this case, however, the equation

$$x^2 (my - px) + k^2 (px - y) = 0$$

shows the asymptotes $x = \frac{k}{\sqrt{m}}$ and $y = \frac{p}{m} x$.

Now, if a plateau DC with a slope OD is given (fig. 13), the X-axis is laid along OB and the Y-axis is drawn at right angles with OB in O. DA is drawn, parallel to OY. Following the nomographical rules for constructing the projective scale $s = \frac{h^2 - my^2}{h^2 - y^2}$ on OB (S-axis), we get

$$\text{for} \begin{cases} y = 0 \;, s = 1 \\ y = \infty, s = m \end{cases} \text{and for} \begin{cases} y = \frac{h}{\sqrt{m}}, & s = 0 \\ y = h & , s = \infty. \end{cases}$$

Choosing OA as unit, the point A is numbered o ($s = 1$). If $OB = m$, the point B is numbered ∞. The origin ($s = o$) is numbered $\frac{h}{\sqrt{m}}$. These three points are sufficient to construct the projective scale. Let the zero-point of the quadratic y-scale coincide with the zero-point A. This quadratic scale may be drawn in any direction and with any modulus (unit). In our case, the scale is laid along AD with a modulus of 1 mm (if $AD = h =$ $= 100$ mm) and numbered (the distance $A - 1 = 1$ mm; $A - 2 = 4$ mm; $A - 3 = 9$ mm etc.). Now, these points of division 1, 2, 3 etc. of the quadratic scale should be projected on the S-axis OB from a centre C to find the scale $s = \frac{h^2 - my^2}{h^2 - y^2}$. The centre C is the point of intersection of two lines, joining similarly numbered points of both scales OB and AD. The point at infinity of AD corresponds to B (∞), so we draw BC parallel to AD ($OB = s = m$). The origin, numbered $\frac{h}{\sqrt{m}}$ should be joined to the point numbered $\frac{h}{\sqrt{m}}$ of the quadratic scale, but it is easier to join the point D of the quadratic scale ($y = h$) to the point at infinity of OB, for we have

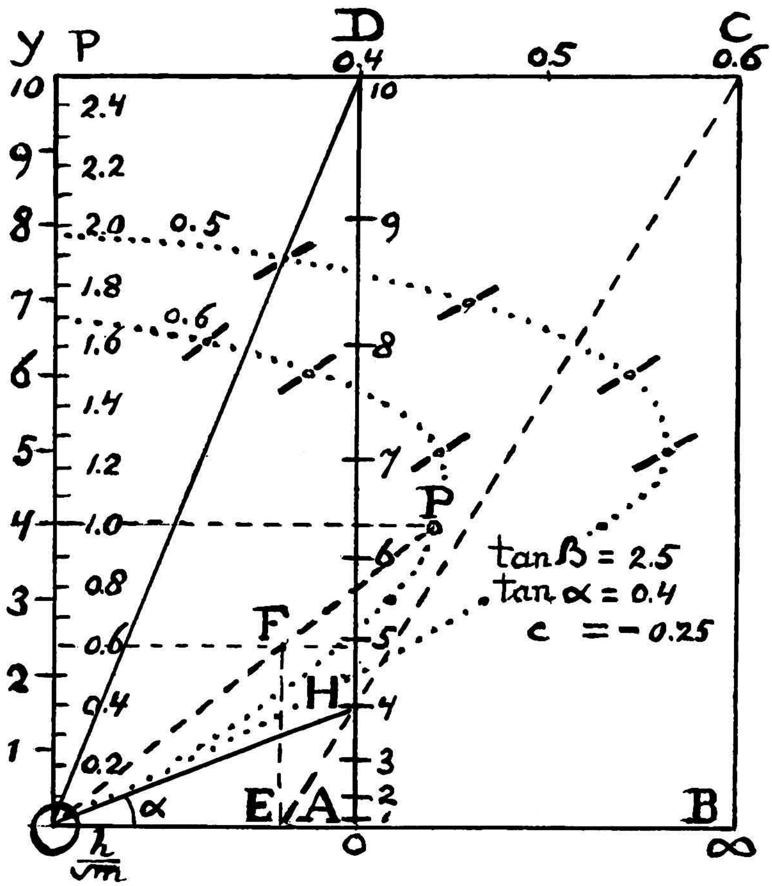

Fig. 13

found before, that for $y = h$, $s = \infty$. The place of the centre C depends upon the value of m and therefore of p. *Thus, each isocline has its own centre of projection C*, numbered as the isocline ($p = 0{,}6$ in fig. 13).

As we have proposed sub 2, the isoclines will be drawn for p-values from $\tan\beta$ to $\tan\alpha$, decreasing by 0,1. The required p-scale has been constructed on the right side of OY by dividing $OY = AD = \tan\beta$ ($OA = 1$) in 25 equal parts (for a value of $\tan\beta = 2{,}5$).

To construct a point P of isocline 0,6 we proceed as follows (fig. 13). The central projection of any point H of the quadratic scale, numbered y_0 from C on OB is E. Now $OE = \frac{h^2 - my_0^2}{h^2 - y_0^2} = s$. The perpendicular in E

meets the line $p = 0{,}6$ parallel to OB in F, so, the direction of $OF = \frac{0{,}6}{s}$. The line $y = y_0 = 4$ meets OF in the desired point $P(x_0, y_0)$ for $\frac{0{,}6}{s} = \frac{y_0}{x_0}$. Repeating the same construction for $y_0 = 1, 2, 3$ etc. we get the dotted curve 0,6, locus of line elements of the integral-curves with direction $\frac{dy}{dx} = p = 0{,}6$. In like manner the isocline 0,5 is constructed by means of the centre 0,5.

As the modulus of the quadratic scale on AD has no influence upon the projective scale on OA, the same construction may be applied to find $s = \frac{k^2 - mx^2}{k^2 - x^2}$. But now we must take the point of intersection of OF with $x = x_0$. The modulus of the X-scale is $0{,}1\,k$.

In fig. 13, $m = 2\,(1 - c)\,(ap - 1) + 1 = 2{,}25$; $a = 2{,}5$; $c = -\,0{,}25$ for $p = 0{,}6$.

4. *Construction of the integral-curve, determined by*

$$x = 0,\ y = 0,\ \frac{dy}{dx} = \tan \beta.$$

a. Case of a plateau.

In fig. 14 an example is given of the construction of the integral-curve of the equation

$$y\,(1 - c)\,(a\,dy - dx) = \frac{h^2 - y^2}{2\,y^2}\,(y\,dx - x\,dy)$$

determined by $x = o$, $y = o$, $\frac{dy}{dx} = 1$, by means of a family of isoclines $px = y\,\frac{h^2 - my^2}{h^2 - y^2}$.

The data are $\beta = 45°$; $\alpha \sim 22°$; $\tan \alpha = 0{,}4$; $a = \cot \alpha = 2{,}5$; $c = 0$; m (for $p = 0{,}6) = YC = 2$. The centra 0,6; 0,55; 0,5 etc. on the line YC refer to the isoclines 0,6; 0,55; 0,5 etc. (distances from point O: 2; 1,75; 1,5 etc.). To construct point P (3) of isocline 0,6, the centre C (0,6) is joined to point 3 of the quadratic scale. The perpendicular in A meets the line $p = 0{,}6$ parallel to OA in B. The line OB meets the line $y = 3$ in the desired point P (see sub 3). In this way, isocline 0,6 and the other isoclines are drawn. To construct the integral-curve we proceed as follows. The line-element in O with the direction $\frac{dy}{dx} = p = 1$ meets the first isocline 0,9 in N. The line-element in N with the direction $\frac{dy}{dx} = 0{,}9$ meets the following isocline 0,8 in L. The line-element in L with the direction 0,8 meets the isocline 0,7 in K etc. We obtain a curved part, hidden underneath the screes and a flat part (approximately beginning with isocline 0,45), where the screes no longer protects the rocky nucleus, so that a further softening may be assumed (16, p. 1155 — 1156 and fig. 7).

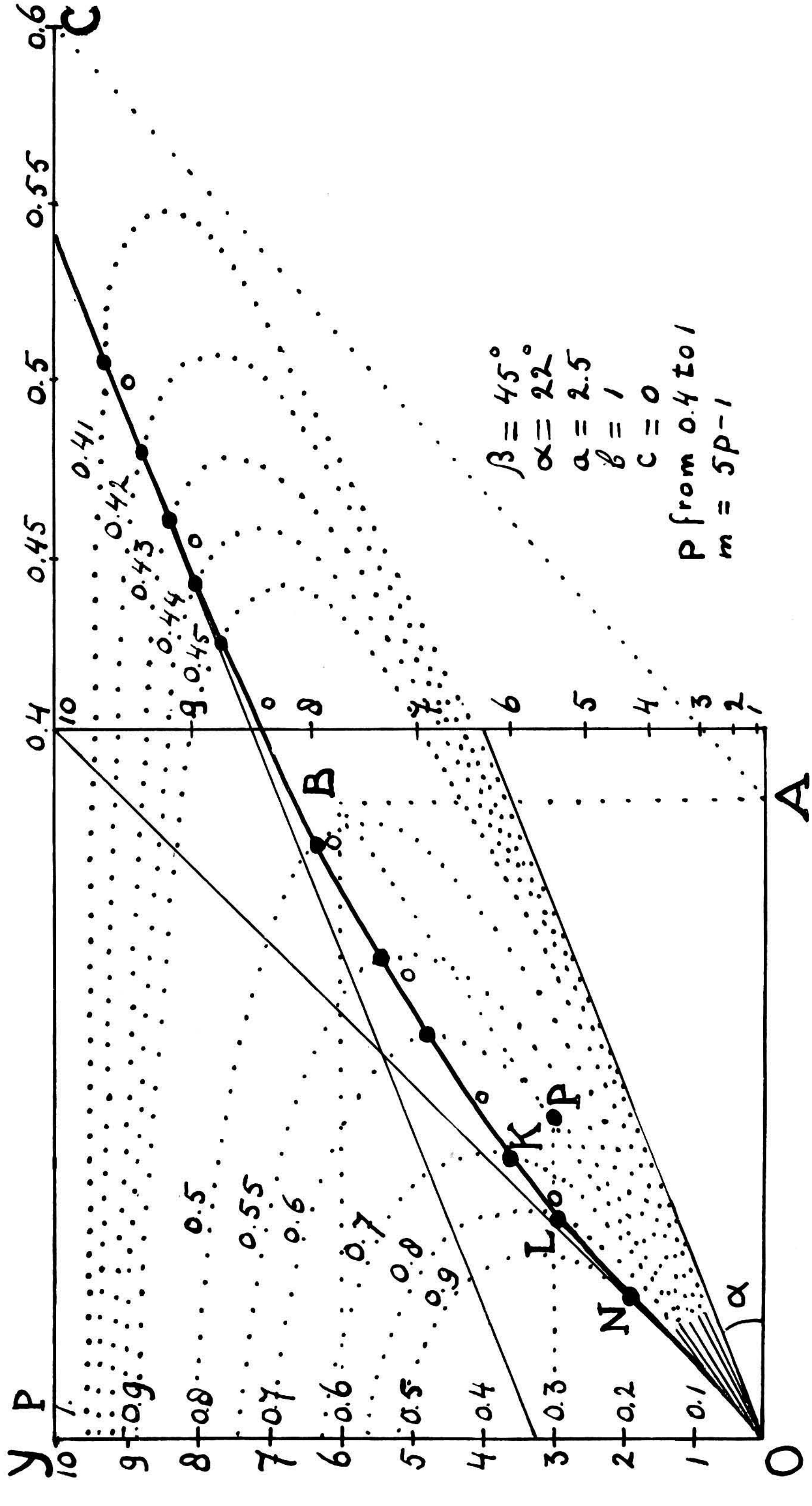

Fig. 14

b. Control by the results, found before.

With the same data, the equation of the integral curve, found before (16, p. 963)

$$x = ay - (a-b)\, y \left\{ \frac{h^2 + (1-2c)\, y^2}{h^2} \right\}^{\frac{c-1}{1-2c}} \tag{14}$$

becomes $x = 2{,}5\, y - 1{,}5 \dfrac{y}{1+y^2}$.

From this equation, we find by calculation the encircled points of fig. 14. Our curve runs nearer to these points, when the interval between the isoclines is smaller.

c. Case of a crest.

In fig. 15 an example is given of the construction of the integral-curve of the equation

$$y(1-c)(a\,dy - dx) = \frac{k^2 - x^2}{2\,x^2}(y\,dx - x\,dy)$$

determined by $x = o$, $y = 0$, $\dfrac{dy}{dx} = 1$ by means of a family of isoclines

$$px = \frac{k^2 - mx^2}{k^2 - x^2}.$$

The data are the same as for fig. 14.

To find point P ($x_0 = 7$) of isocline 0,45, the centre of projection C ($m = 5p - 1 = 1{,}25$; $p = 0{,}45$) is joined to point 7 of the quadratic

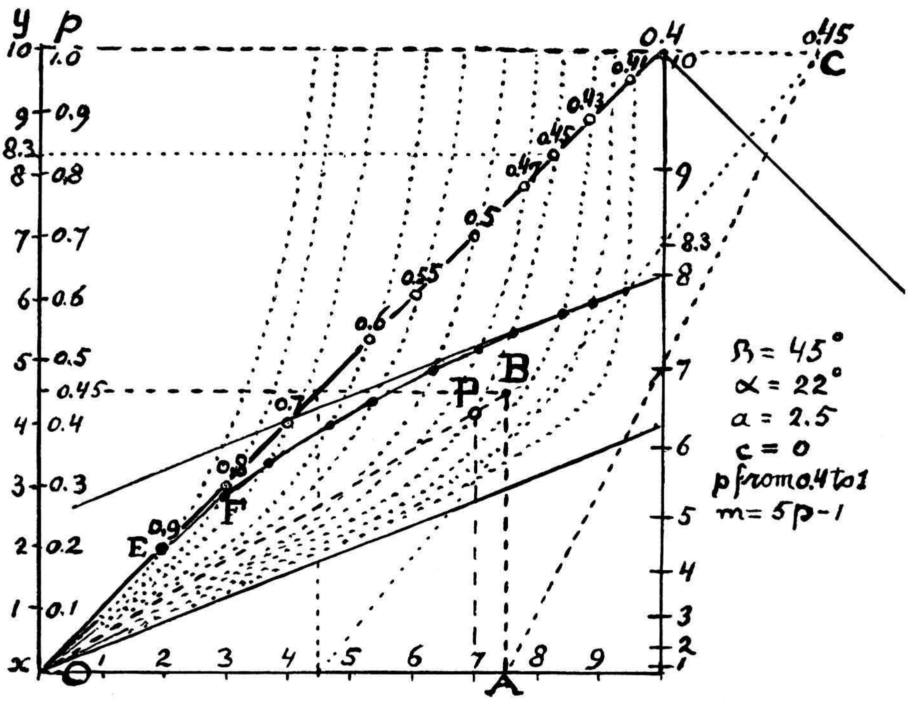

Fig. 15

scale. This line meets the S-scale in A. The perpendicular in A meets the line $p = 0{,}45$ parallel to OA in B. The line $x = x_0 = 7$ meets the line OB in the desired point P. Repeating this construction for several points of the quadratic scale, we get isocline 0,45. Repeating it again for centra on the line YC, we get a family of isoclines.

The integral-curve is constructed in the same way as sub a. The line-element in O with the direction $\frac{dy}{dx} = p = 1$ meets the first isocline 0,9 in E. The line-element in E with direction 0,9 meets isocline 0,8 in F etc. In the case of a crest, the rock-profile has a somewhat greater curvature, due to the vertical direction of the isoclines.

5. *Simplification of the construction.*

Although it is now possible to construct as many points of the isoclines as we desire, practical use requires a more rapid construction of the integral-curve and a conception of the ratio between the flat part and the curved part.

In the first place, we only want some two or three points of each isocline near the probable course of the integral-curve.

In the second place, we can sketch the general course of an isocline, if we bear in mind, that each curve has a tangent $y = px$ in the origin O and if we know some special points.

We have already remarked (sub 3), that the centre C must lie upon the line, joining O (numbered $\frac{h}{\sqrt{m}}$ in the case of a plateau and $\frac{k}{\sqrt{m}}$ in the case of a crest) to the similarly numbered point of the quadratic scale.

Reversely: joining O to the centra of all isoclines, the values $\frac{h}{\sqrt{m}}$ or $\frac{k}{\sqrt{m}}$ are read off upon the quadratic scale. But $\frac{h}{\sqrt{m}}$ is the ordinate of the point where isocline $px = y\,\frac{h^2-my^2}{h^2-y^2}$ ... (18) meets the Y-axis (putting $x = o$) and $\frac{k}{\sqrt{m}}$ is the abscissa of the point where the asymptote of $px = y\,\frac{k^2-mx^2}{k^2-x^2}$ (19) parallel to OY meets the X-axis. Thus, these values are immediately found.

In fig. 16 isocline 0,5 (plateau) will meet the Y-axis in A (7,5) and isocline 0,5 (crest) has an asymptote $x = 7{,}5$ (meets YH in D). The tangent in O is OG for both isoclines.

The meeting-points of the isoclines with the initial slope are also easily to be found in both cases. From the construction of point 6,8 of isocline 0,5 (fig. 16) it follows, that the perpendicular in E meets the line $p = 0{,}5$ parallel to OP just upon the initial slope OH. Consequently, the desired point B must also lie on the slope and therefore, it is the meeting point of isocline 0,5 with the slope. Reversely: to find the ordinates of the meeting points of all isoclines with the initial slope, we must proceed as follows. Draw the lines 0,5; 0,6; 0,7 etc. parallel to OP till they meet the slope. From these points of intersection, let the perpendiculars fall upon OP and join the feet E etc. to the centra 0,5; 0,6; 0,7 etc. The numbers of the meeting-points of these

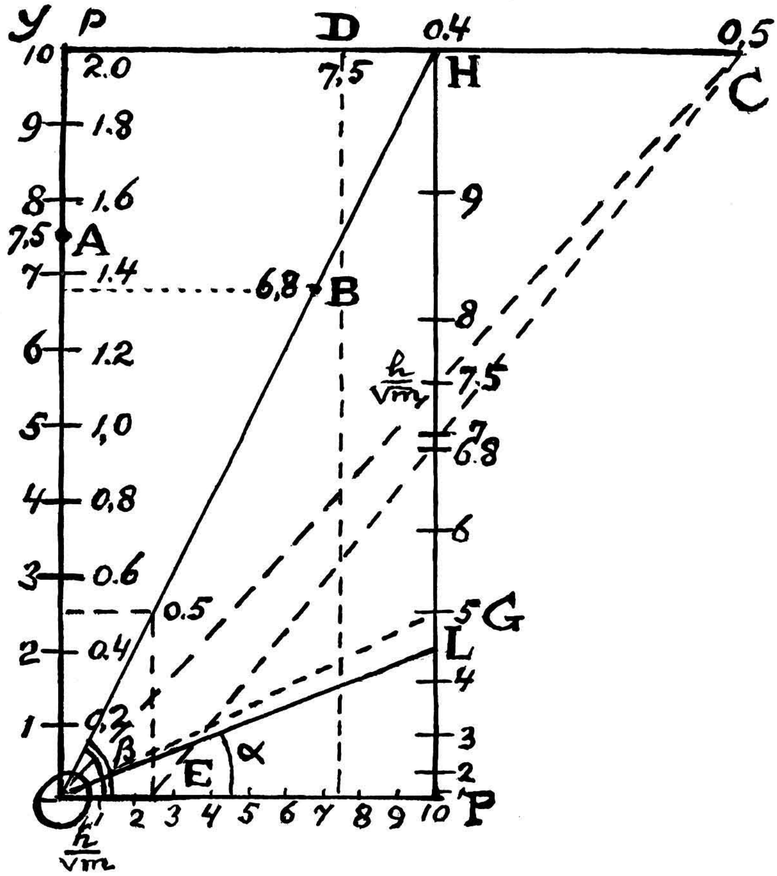

Fig. 16

lines with the quadratic scale PH will give the ordinates of the points of intersection of the isoclines with the initial slope. It must be noted, that, with the same data, we find the *same* meeting-points, whether the profile is that of a plateau or that of a crest. For, the equation of the slope being $y : h = x : k$, the equations

$$px = y\,\frac{h^2 - my^2}{h^2 - y^2} \text{ and } px = y\,\frac{k^2 - mx^2}{k^2 - x^2}$$

will give the same values of x and y.

Examples:

The slope in fig. 14 has the same scale of intersection-points as the slope in fig. 15. In fig. 17*a*, O has been joined to the centra of the isoclines

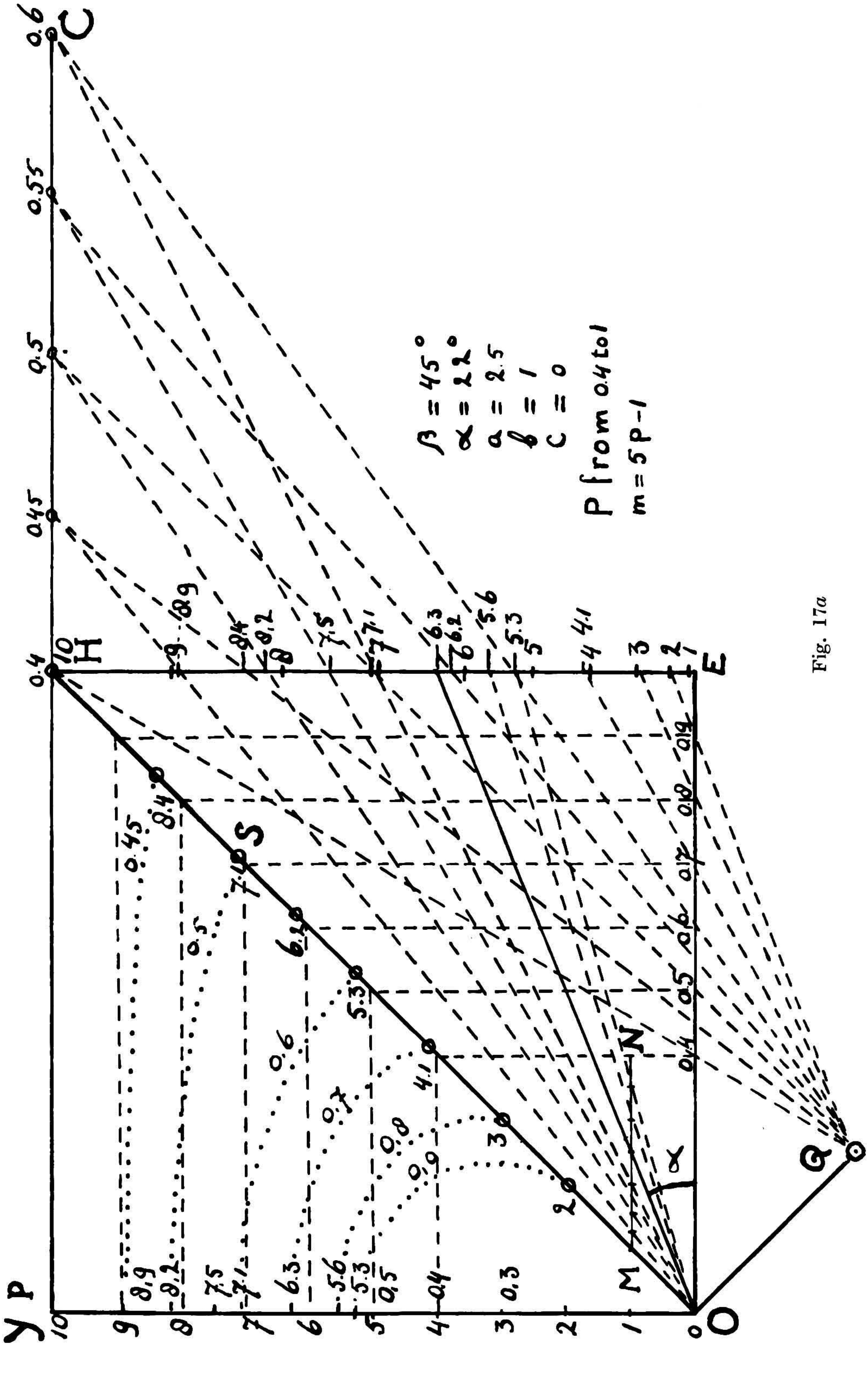

Fig. 17a

0,45; 0,5; 0,55; 0,6; 0,7; 0,8 and 0,9 (auxiliary-scale *MN* 1 : 10). The quadratic scale *HE* shows, that the isoclines will meet the *Y*-axis in points with ordinates 8,9; 8,2; 7,5; 7,1; 6,3; 5,6 and 5,3.

To find the points where the isoclines meet the slope, the feet of the perpendiculars should be joined to the corresponding centres. But these lines, joining up similarly numbered points of two parallel, regular scales, will pass through a fixed point *Q*. Making use of this point, the regular scale of the centra may be omitted and the figure for constructing the points of intersection on the slope remains limited between the parallels *OY* and *EH*. By joining *Q* to the points of division upon *OE*, we find the ordinates 8,4; 7,1 etc. of the meeting-points of the isoclines with the slope.

By making use of the new guise of our theory, we shall present the problem of the ratio between the flat part and the curved one of the rocky nucleus underneath the screes in the fourth part of our treatment.