MATHEMATICS

COMPACT SPACES WITH A LOCAL STRUCTURE DETERMINED BY THE GROUP OF SIMILARITY TRANSFORMATIONS IN E^n

BY

N. H. KUIPER

(Communicated by Prof. W. VAN DER WOUDE at the meeting of June 24, 1950)

§ 1. *The problem.*

The spaces X we want to consider are n-dimensional manifolds with a local structure which we shall define with the help of the similarity space Es^n as a fixed reference space. Es^n is the Euclidean n-dimensional space assigned with the group I of similarity transformations ("similarities") i.e. the group generated by the Euclidean motions and the geometrical multiplications.

A *reference system* is a topological mapping Φ of a (open) domain V in X onto a domain $\Phi(V)$ in Es^n. It carries the local structure (e.g. angles, straight lines) of Es^n in V. Two reference systems (Φ, V) and (ψ, W) are said to agree, if for every component U of the intersection $V \cap W$, the topological mapping $\psi\Phi^{-1}: \Phi U \rightarrow \psi U$ is a similarity in Es^n. If all reference systems of a set agree mutually, then they determine a unique local structure in the covered pointset of X. The set is then called *a set of preferred reference systems*. Any additional reference system is preferred if it agrees with all others. A set of preferred reference systems, covering X, is called complete if it contains any reference system that agrees with it. We now define: An n-dimensional manifold X is called (*locally*) *sim E* [1]), if it is covered by a complete set of preferred reference systems, the reference being with respect to Es^n.

Our problem is analogous to the space problem of CLIFFORD—KLEIN ([2], [4]) and consists in *the examination of all compact* (= X is covered by a finite subset of any set of neighbourhoods that cover X.) *connected locally sim E spaces* X. The results are stated in the theorem at the end of the paper.

§ 2. *Preliminaries.*

If X is sim E and connected, then so is the universal covering space $\tilde{X}$ of X. It is well known that in $\tilde{X}$ a discrete group of topological transformations without fixed point, the covering group D, operates. Any $d \in D$ leaves the sim E structure of $\tilde{X}$(!) invariant. A point of X can be

[1]) In a theory of spaces with generalised displacements, the name "similarity-flat" would be appropriate.

considered as a point set $D\tilde{x}(\tilde{x} \in \tilde{X})$ of $\tilde{X}$. D is isomorphic to the fundamental or POINCARÉ group of X.

For simply connected sim E spaces, e.g. $\tilde{X}$, the following theorem holds.

Lemma 1. If $\tilde{X}$ is simply connected and sim E, then a mapping $\Phi : \tilde{X} \to \tilde{X}^* \subset Es^n$ exists, such that any point $\tilde{x} \in \tilde{X}$ has a neighborhood $N(\tilde{x})$ for which $(\Phi, N(\tilde{x}))$ is a preferred reference system. Any two such mappings Φ, ψ obey a relation $\Phi = g \cdot \psi$, where $g \in I$ is a similarity.

We omit the proof of this lemma which is a consequence of an analogous theorem on conformally flat spaces (KUIPER [5], th. 4; analogous theorems with analogous proofs hold for a large class of spaces, the locally homogeneous spaces [3], [7]).

If Φ is fixed according to lemma 1, and $d \in D$, then Φd is another mapping with the same properties as Φ. Hence $d^* \in I$ exists such that $\Phi d = d^* \Phi$ (lemma 1). It is easily seen that the correspondence $d \to d^*$ is a homomorphism of D onto a subgroup $D^* \subset I$. If $\Phi : \tilde{X} \to \tilde{X}^*$ is topological then D and D^* are isomorphic.

From now on X denotes a compact connected sim E space. X then admits a finite triangulation, and this yields in the natural way a triangulation of $\tilde{X}$. It is possible to choose an open *fundamental domain* F in $\tilde{X}$, composed of simplices of the triangulation, with the following properties: The covering mapping $f : \tilde{X} \to X$, restricted to F, is topological $f : F \longleftrightarrow f(F) \subset \subset X$; $f(\bar{F}) = X$ ($\bar{F}$ is the closure in $\tilde{X}$ of F); the boundary $\bar{F} - F$, as well as its image $f(\bar{F} - F)$ is the union of a finite number of closed n—1-dimensional simplices, called faces: If d_1 and d_2 are different elements of the covering group D, carrying F onto the fundamental domains $d_1(F)$ and $d_2(F)$ respectively, then $d_1(F) \cap d_2(F)$ is void: However the union of all $d(\bar{F})$ for $d \in D$ is $\tilde{X}$. From now on F denotes a fixed fundamental domain, and Φ or (*) a fixed mapping according to lemma 1.

§ 3. *Φ has no boundary points.*

Definition: A boundary point of $\Phi : \tilde{X} \to \tilde{X}^* \subset Es^n$ is the end point $\Phi(0)$ of a continuous curve $\Phi(t)$, $0 \leqq t \leqq 1$, in Es^n with the property: There exists a curve $\tilde{x}(t) \subset \tilde{X}$ such that $\Phi(\tilde{x}(t)) = \tilde{x}^*(t) = \Phi(t)$ for $0 < t \leqq 1$; there does not exist a point $\tilde{x}(0) \subset \tilde{X}$, such that moreover $\tilde{x}(t)$ is a continuous curve for $0 \leqq t \leqq 1$.

If Φ has no boundary points then: $\tilde{X}^* = Es^n$; $(\Phi, \tilde{X})$ is a covering-space (in the topological sense) of $\tilde{X}^* = Es^n$; and as Es^n is simplyconnected, Φ is a topological mapping and D and D^* are isomorphic. Suppose $d^* \in D^*$ is a similarity transformation which is not a Euclidean motion. Then d^* has a "factor of multiplication of distances" $f < 1$ (if $f > 1$, we take $(d^*)^{-1}$), and an invariant point, say A, in Es^n. Let $d \in D$ correspond with d^* under the isomorphism $D \longleftrightarrow D^*$; Let $\tilde{x}(t)$, $f \leqq t \leqq 1$, be a continuous curve with endpoints $\tilde{x}(1)$ and $\tilde{x}(f) = d\,\tilde{x}(1)$. Finally let $\tilde{x}(t)$, $0 < t \leqq 1$, be defined by

$$\tilde{x}(t) = d^k \cdot \tilde{x}(f^{-k} \cdot t) \quad \text{for} \quad f^{k+1} \leqq t \leqq f^k, \ k = 1, 2, \ldots$$

Then $\lim_{t \to 0} \Phi(\tilde{x}(t)) = A$ is a boundary point in contradiction with the assumptions. Hence D^* contains only motions. Therefore it is possible to introduce a locally Euclidean metric in X, and we have:

Lemma 2. If X is compact and sim E, and $\Phi : \tilde{X} \leftarrow \tilde{X}^*$ has no boundary points, then Φ is topological, and X can be considered as a compact locally Euclidean space of which only the local sim E structure is taken into account.

§ 4. *Boundary points of Φ.*

Let B be a boundary point of $\Phi : \tilde{X} \to \tilde{X}^* \subset Es^n$, defined by $\tilde{x}(t)$ and $\Phi(\tilde{x}(t)) = \Phi(t)$, $0 < t \leqq 1$, as above. Suppose $\tilde{x}(t)$ meets for t running from 1 to 0 successively the closed fundamental domains $d_\nu(\overline{F})$, $\nu = 1, 2, \ldots$. If this were only a finite number of pointsets, with a compact sum (!), then a point $\tilde{x}_0 \in \tilde{X}$ would exist, in any neighborhood of which points $\tilde{x}(t)$, $t < \varepsilon$, $\varepsilon > 0$ and arbitrary, would occur. The inverse of the topological mapping Φ of some neighborhood of $\tilde{x}_0$ in Es^n, then would show that B is not a boundary point with respect to $\tilde{x}(t)$. Contradiction. Hence $\tilde{x}(t)$ meets an infinite number of closed fundamental domains $d_\nu(\overline{F})$. Also if $\varepsilon > 0$, then a number $N(\varepsilon)$ exists, and $\tilde{x}(t)$ respectively $\Phi(t)$, $t < \varepsilon$, does meet the pointset $d_\nu(\overline{F} - F)$ respectively $\Phi d_\nu(\overline{F} - F) = d^*_\nu(\overline{F}^* - F^*)$ for all $\nu > N(\varepsilon)$. (1)

Similarities g of which the factor of multiplication of distances, denoted by $|g|$, is bounded: $0 < p \leqq |g| \leqq q < \infty$, p, q constants, form a compact subset of the group space I. Hence if the factors $|d^*_{\nu}|$ are bounded in this way, then some subsequence $d^*_{\nu(\mu)}$ of d^*_ν will converge to a similarity say d^*. If μ is large, then $d^*_{\nu(\mu)}$ transforms some point of $\Phi(\overline{F}) = \overline{F}^*$ into a point close to B. Without restriction we may then assume F to be such that B lies in the interior of $d^*(F^*)$. This implies that B has a neighborhood which does not contain any of the points of $d^*_{\nu(\mu)}(\overline{F}^* - F^*)$ for μ sufficiently large; in contradiction with (1).

Because the factors $|d^*_\nu|$ are not bounded, a subsequence $d_{\nu(\mu)}$ of d_ν will exist for which $\lim_{\mu \to \infty} |d^*_{\nu(\mu)}| = \infty$ or $= 0$. In the first case $\lim_{\mu \to \infty} |d^*_{\nu(\mu)})^{-1}| = 0$. If μ is large, then $(d^*_{\nu(\mu)})^{-1}$ transforms some point very near to B (in the intersection of $\Phi(t)$ and $d^*_{\nu(\mu)}\overline{F}(!)$) into $\overline{F}^*$. Because moreover $|(d^*_{\nu(\mu)})^{-1}|$ is very small, $(d^*_{\nu(\mu)})^{-1} \cdot \Phi(t)$ will prove to have a limit point for $t \to 0$, which is not a boundary point with respect to $(d^*_{\nu(\mu)})^{-1} \cdot \tilde{x}(t)$. But then $\lim_{t \to 0} \Phi(t) = B$ is not a boundary point with respect to $\tilde{x}(t)$ either.

Finally we come to the conclusion that d^*_ν contains a subsequence $d^*_{\nu(\mu)}$, with factors, that tend to zero. Because moreover the pointsets $d_\nu^*(F^*)$ contain points that tend to B, the invariant points of the similarities $d_{\nu(\mu)}^*$ also tend to B. This proves

Lemma 3. If B is a boundary point, $N(B)$ a neighborhood of B, U a bounded pointset in Es^n, then D^* contains an element d^*, such that $d^*(U) \subset N(B)$.

§ 5. *The set of boundary points.*

Lemma 4. *Let V be a closed subset of the Euclidean space E^n, not contained in any E^{n-1}; $n \geqq 0$. Let G be a subgroup of the similarity group operating in E^n, such that for any $g \in G : gV = V$.*

Moreover suppose that for any point B with neighborhood $N(B)$ in V and any bounded pointset U in E^n, there exists an element $g \in G$ such that $g \cdot U \subset N(B)$.

Then $V = E^n$.

(Roughly: small parts of V are similar to large parts).

Proof:

The lemma is trivial for $n = 0$. We suppose $n > 0$.

First we agree that any neighborhood under consideration will be the interior of a hyper sphere. Let $U(A)$ and $N(A) \subset U(A)$ $(N(A) \neq U(A))$ be neighborhoods of the point $A \in V$. Let $g \in G$ be such that $g \cdot N(A) \subset \subset g \cdot U(A) \subset N(A)$. Straightforward geometrical consideration as well as the BROUWER fixed point theorem then show that $N(A)$ contains a point A^* invariant under g. Let the factor of multiplication of distances of g be $|g| < 1$. The set of points $g^m \cdot A$ $(m = 1, 2, \ldots)$ all belonging to the closed set V, converges to A^*, which therefore also belongs to V.

Let $|g|$ also denote the geometrical multiplication with centre A^* and with the same factor of multiplication of distances as g. $h = g \cdot |g|^{-1}$ is then a rotation about A^*. Suppose the point $B \neq A$ is in $V \cap U(A)$. The set of points $h^m B (m = 1, 2, \ldots)$ has at least one limit point on the hypersphere with centre A^* and radius $=$ distance (A^*, B). Therefore if $\varepsilon > 0$, integers m_1 and $m_2 = m_1 + m > m_1$ exist, for which

$$\text{angle } (h^{m_1} B, A^*, h^{m_2} B) = \text{angle } (B, A^*, h^m B) = \text{angle } (B, A^*, g^m B) < \varepsilon.$$

Because $g^m \cdot B = C$ is a point of V as well as of $N(A)$ we have:

Statement 1. If A and B are different points in V, $N(A)$ is a neighborhood of A, $\varepsilon > 0$, *then* there exist points A^* and C in $V \cap N(A)$, for which angle $(B, A^*, C) < \varepsilon$.

This statement allows us to verify:

Statement 2. There exists a set of points $A_1, A_2^*, A_3^*, \ldots, A_{n+1}^*$, R, all in V, such that R is interior point of the nondegenerated simplex (Euclidean, not only topological) with the other points as vertices.

The existence of $n + 1$ points $A_1, A_2, \ldots, A_{n+1}$ in V, but not in any E^{n-1}, follows from the conditions in the lemma. We replace A_2 by two points A_2^* and C_1 both in V, near to A_2 and such that the angle (A_1, A_2^*, C_1) is so small that C_1 lies on the same side of the hyperplanes spanned by $A_2^*, A_3, A_4, \ldots, A_{n+1}$ and by $A_1, A_3, \ldots, A_{n+1}$ as the inside of the simplex with vertices $A_1, A_2^*, A_3, \ldots, A_{n+1}$ does. Statement 1 allows this choice.

In the next step we replace A_3 and C_1 by two points A_3^* and C_2 in V, near to A_3, and such that C_2 lies on the same side of the hyperplanes $(A_2^*, A_3^*, A_4, \ldots, A_{n+1})$, $(A_1, A_3^*, A_4, \ldots, A_{n+1})$ and $(A_1, A_2^*, A_4, \ldots, A_{n+1})$ as the inside of the simplex $(A_1, A_2^*, A_3^*, A_4, \ldots, A_{n+1})$. Statement 1 allows this choice.

Continuing with analogous steps we finally obtain the required set of points $A_1, A_2^*, A_3^*, \ldots, A_{n+1}^*$ and $C_n = R$.

Let $C(T, \psi)$ be a solid hypercone in E^n, that is the locus of halflines which in their fixed endpoint T meet one of these half lines under angles $\leqq \psi < \pi/2$. Let us denote the part of a $C(T, \psi)$ which is between or in two concentric hyper spheres with centre T and radii r and $r \cdot f (r > 0, 0 < f \leqq 1)$ by $CS(T, \psi, f)$. Then we are able to formulate:

Statement 3. There exists a point $R \in V$, an angle $\psi < \pi/2$ and a number $f > 0$, such that any $CS(R, \psi, f)$ contains at least one point of V.

Consider a configuration $A_1, A_2, \ldots, A_{n+1}, R$ of points in V, for which R lies in the interior of the simplex $A_1, \ldots, A_{n+1}$. Then there exists a number $\psi < \pi/2$, such that any solid cone $C(R, \psi)$ contains at least one of the points $A_1, \ldots, A_{n+1}$. Even ψ can be choosen such that the same is true if we replace R by any point in a sufficiently small neighborhood $N(R)$ of R. So we do. Now let $g \in G$ transform a large neighborhood $U(R)$ of R onto $g \cdot U(R) \subset N(R)$. g has an invariant point in $N(R) \cap V$. Without restriction we may assume that this point is R. Let $|g| < 1$ be the factor of multiplication of distances of g, and let $p \leqq 1$ be the ratio of the minimum and the maximum of the distances $(A_1, R), (A_2, R), \ldots, (A_{n+1}, R)$. Then the property of statement 3 holds for any

$$CS(R, \psi, f) = CS(R, \psi, |g| \cdot p/2)$$

(Consider the set of points $g^m A_i \subset V$; $m = 1, \ldots; i = 1, \ldots, n+1$.)

Statement 4. If SS is a solid hyper sphere in E^n, $\psi < \pi/2$, $0 < f \leqq 1$, then for any point P sufficiently near to SS, there exists a $CS(P, \psi, f)$ which is completely contained in the interior of SS.

This follows immediately from the analogous statement for the Euclidean plane E^2.

Assumption. Now suppose $V \neq E^n$. Then because V is closed, there exists a solid sphere SS in E^n, the inside of which has no point in common with V, but the boundary of which intersects V in at least one point T. (V is not void by assumption !). There exists a transformation $h \in G$, which transforms the point R of statement 3 into a point so near to the point T of the boundary of SS, that there exists a $CS(hR, \psi, f)$ completely contained in the interior of SS (Conditions in the lemma; statement 4;

ψ and f are the constants obtained in the proof of statement 3). That particular $CS(hR, \psi, f)$ *does not contain any point of* V, because the interior of SS does not. On the other hand h is a $1 \longleftrightarrow 1$ similarity transformation which transforms all $CS(R, \psi, f)$ onto all $CS(hR, \psi, f)$ and which leaves V invariant. Because all $CS(R, \psi, f)$ did contain points of V, so do *all* $CS(hR, \psi, f)$. Hence the assumption leads to a contradiction, and $V = E^n$ q.e.d.

As a corollary of lemma 4 we have

Lemma 5.

Let V be a closed subset of E^n; $n \geqq 0$. Let G be a subgroup of the similarity group operating in E^n, and for any $g \in G : gV = V$. Moreover suppose that for any point B in V with neighborhood $N(B)$ and any bounded pointset U in E^n, there exists a $g \in G$ for which $g \cdot U \subset N(B)$.

Then $V = E^m$, $m = -1, 0, 1, \ldots,$ or n.

Proof: Apply lemma 4 to the linear space E^m of smallest dimension which contains V.

Lemma 5 can be applied to our problem: The closure of the set of boundary points in Es^n of the fixed mapping Φ, is invariant under the similarity transformations $d^* \in D^*$, operating in Es^n. Lemma 3 presents the other necessary conditions for application of lemma 5. Hence:

Lemma 6. *The closure of the set of boundary points in* Es^n, *under the fixed mapping* $\Phi : \tilde{X} \to \tilde{X}^*$, *is a linear space* E^m, $m = -1, 0, 1, \ldots, n$.

§ 6. *The set of boundary points. Continued.*

Assumption 1. In lemma 6 is the number $m = n$.

Let A be the invariant point of the similarity $a \in D^*$ with factor of multiplication of distances $|a| < 1$. Let $\tilde{x}^* \in \tilde{X}^*$ and let the pointset $\Phi^{-1}(\tilde{x}^*)$ consist of the points $\tilde{x}_\lambda (\lambda = 1, 2, \ldots)$. Let, under the homomorphism $D \to D^*$, d correspond with a. As in § 3 we construct with *each* point $\tilde{x}_\lambda$ as initial point $\tilde{x}_\lambda(1)$, a continuous curve $\tilde{x}_\lambda(t)$ $(0 < t \leqq 1)$ for which $\lim_{t \to 0} \Phi(\tilde{x}_\lambda(t)) = A$ is a boundary point! From that it follows that $\Phi^{-1}(A)$ is void. The set of invariant points like A is dense in E^n (consequence of assumption 1). Assumption 1 leads to the contradiction: $\tilde{X}^*$ is void.

Assumption 2: In lemma 6: $0 < m < n$.

Let A and B be two different invariant points of similarities a and b respectively; $a, b \in D^*$, $|a| < 1$, $|b| < 1$. Consider the set of numbers $|a|^r \cdot |b|^s$ ($r \neq 0$, $s \neq 0$ integers). This set has the number 1 as a limit point. Therefore a sequence $c_\nu = a^{r_\nu} b^{s_\nu} \in D^*$ exists for which $\lim_{\nu \to \infty} |c_\nu| = 1$. From the activity of c_ν on the point B we deduce that the set $(c_\nu)^\mu (\nu = 1, \ldots; \mu =$ integer $\mu \neq 0)$ has in the similarity group space the identity *not* as an accumulation point.

Let ε be a small positive number. We determine a number $N = N(\varepsilon)$

so large that for any rotation $\mathfrak{r}$ in a centered Euclidean vector space of dimension n and a vector $\vec{v}$, the angle between $\vec{v}$ and at least one of the vectors $\mathfrak{r}^{\mu}\vec{v}$, $0 < \mu < N$, is smaller then ε. This is possible because the sphere of unit vectors is compact and has an invariant measure (angle).

Next we choose $c = a^r \cdot b^s$ obeying (c exists!)

$$1 - \varepsilon < |c^{\mu}| = |a^{\nu}c^{\mu}a^{-\nu}| < 1 + \varepsilon \text{ for all } 0 < \mu < N, \ \nu \text{ integer.}$$

We denote the set of boundary points of Φ by E^m. $A, B \in E^m$. Let the line (P^*, A) be perpendicular to E^m, and let $P \in \tilde{X}$, $\Phi(P) = P^* \in \tilde{X}^*$. The points $(a^{\nu} c^{\mu} a^{-\nu}) A$ are different from A, though for $\nu \to \infty$ they converge to A. There exists a value $\nu = p$, so that the distance $(A - (a^p c^{\mu} a^{-p})A)$ is smaller then ε for $0 < \mu < N$. Idem for any $\nu < p$.

The points $(a^p c^{\mu} a^{-p}) P^*$ $(0 < \mu < N)$ are contained in the linear space of dimension $n - m$ perpendicular to E^m in the point $(a^p c^{\mu} a^{-p})A$.

The angle between the line $\{A \to P^*\}$ and the line

$$\{(a^p c^{\mu} a^{-p})A \to (a^p c^{\mu} a^{-p})P^*$$

is, for a particular choice: $\mu = u$, smaller then ε (see above).

Because ε was arbitrary, it now follows (elementary) that any neighborhood $N(P^*)$ of P^* contains points $d^* P^*$ $(d^* \in D^*)$ different from P^* and equivalent with P^*.

If $0 < m < n - 2$ or $m = n - 1$, then $\tilde{X}^*$ determined by E^m is simply connected and the mapping $\varphi : \tilde{X} \to \tilde{X}^*$ is topological. The points aequivalent with P^* under the covering group *cannot* converge to P^* in this case. Contradiction.

If $m = n - 2$, then Φ is not topological and $(\Phi, \tilde{X})$ is the universal covering space of $\tilde{X}^*$. Every point $\tilde{x} \in \tilde{X}$ can be described by the point $\tilde{x}^* = \Phi(\tilde{x})$ and an angle of rotation about the E^{n-2} of boundary points. We call the angle: argument. Let under the homomorphism $D \to D^*$: $a \to a$, $c \to c$. Clearly:

$$\arg (a^{\nu} c^{\mu} a^{-\nu}) P = \arg c^{\mu} P.$$

The points $(a^{\nu} c^{\mu} a^{-\nu})P$, $\nu < p$, $0 < \mu < N$, are then different and aequivalent under the covering group D in $\tilde{X}$, and they are contained in a compact bounded set of $\tilde{X}$. Hence they have a limit point in $\tilde{X}$: contradiction.

Assumptions 1 and 2 being false, Lemma 6 yields: $m = -1$ or 0.

$m = -1$ was considered in lemma 2. If $m = 0$, $n > 2$, then Φ is topological; the simplyconnected space $\tilde{X}^*$ can be identified with $\tilde{X}$. If $m = 0$, $n = 2$, then $\tilde{X}$ is the wellknown universal covering space of a plane with exception of a point. If $m = 0$, $n = 1$, $\tilde{X}^*$ is a Euclidean half line. (The case $m = 0$, $n > 2$ was considered in KUIPER [6] section 5*b*). We finally state our results in the

Theorem. *The universal covering space (with preservation of the sim E structure!) of a compact connected locally sim E space of dimension $n > 0$ is either the Euclidean space Es^n, or a) for $n > 2$: the same with exception of one point, b) for $n = 2$: the universal covering space of the Euclidean plane with exception of one point, c) for $n = 1$: the Euclidean half line. In the first case the original space is determined (identification) by a (covering-)group of Euclidean transformations (lemma 2). In the second case a) idem by a group generated by a subgroup Ω of rotations about the excluded point and a similarity b (b is not a rotation) with the excluded point as invariant point, and which commutates with Ω: $\Omega b = b\Omega$. In the second case b) we use polar coordinates: (radius, angle) $= (r, \omega)$ in the space $\tilde{X}$. The covering group is generated by two transformations:* (1) $r' = r$, $\omega' = \omega +$ constant and (2) $r' = K \cdot r$ $(0 < K < 1)$, $\omega' = \omega +$ *constant. In the third case c) the covering group is generated by a geometrical multiplication:* $r' = K \cdot r$ $(0 < K < 1)$.

Technische Hogeschool, Delft, Holland.

REFERENCES

[1] S. Bochner, On compact complex manifolds. J. Ind. Math. Soc. XI (1947).

[2] E. Cartan, Lecons sur la géometrie des espaces de Riemann, Chapitre III et VI (Paris, 1946).

[3] Ehresmann. C., Sur les espaces localement homogenes. Enseign. Math. **35**, 317–333 (1936).

[4] Hopf, H., Zum Clifford-Kleinschen Raumproblem, Math. Annalen **95**, 313–340 (1926).

[5] Kuiper, N. H., On conformally flat spaces in the large. Ann. of Math. **50**, 916–924 (1949).

[6] ———, On compact conformally Euclidean spaces of dimension > 2. Ann. of Math. **51**, (1950).

[7] Whitehead, J. H. C., Locally homogeneous spaces in differential geometry. Ann. of Math. **33**, (1932).