MATHEMATICS

ON CERTAIN LOCAL PROPERTIES OF A TOPOLOGICAL SPACE ASSOCIATED WITH A PSEUDO-METRIC. I

BY

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(Communicated by Prof. B. L. VAN DER WAERDEN at the meeting of June 24, 1950)

1. *Introduction.* In a previous paper [1]) the author has given a new proof of the well-known theorem of POINCARÉ and VOLTERRA that a multiform analytic function assumes at a given point of the complex plane only a countable number of distinct values. He has announced there that he will generalize the reasoning to a theory of local hereditary properties of metric spaces and more generally to that of certain local hereditary properties of uniform spaces. He also made there a short sketch of the chief features of this generalization. It is this programme which shall be carried out in the work bearing the above title and of which the present paper is the first part.

We consider in this Part I not a metric in a metrizable space, but a pseudo-metric and more generally what we call a semi-pseudo-metric in an arbitrary T-space (cf. § 2). The chief rôle is played by what we call *element-invariants*, in particular by those termed basic integers (cf. § 8). It will probably be impossible to find precise set-theoric topological equivalents for the element-invariants. Consequently our theory, though very general and simple, seems to be of purely metric nature.

2. *Local spherically hereditary properties of a T-space R associated with a semi-pseudo-metric in R.* Let R be a T-space (cf. [2, 27], [1, 37] [2])). If to every ordered pair of points x, y of R a non-negative finite real number $f(x, y)$ is assigned such that

1° $f(x, x) = 0$;

2° $f(x, y) = f(y, x)$ (Symmetry axiom);

3° $f(x, z) \leqslant f(x, y) + f(y, z)$ (Triangle axiom);

4° for every $x \in R$ and $\eta > 0$, $\{y \mid f(x, y) < \eta\}$ is an open set;

then the function $f(x, y)$ is termed, following J. W. TUKEY, a *pseudo-metric in R* [2, 50]. A function $f(x, y)$ in R satisfying 1°, 2°, 4° only (but *not* the triangle axiom) will be called a *semi-pseudo-metric in R.*

The set $\{x \mid f(a, x) < \varepsilon\}$ $(\varepsilon > 0)$ is termed the *sphere of radius* ε and

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[1]) "On a theorem of Poincaré and Volterra", accepted for publication by the London Math. Soc. This paper will be referred to in the sequel as P.V.

[2]) References to the bibliography at the end of the paper are given in brackets. The first number designates the entry and subsequent numbers the pages.

of centre a (or *about a*) in the semi-pseudo-metric f in R (cf. [2, 51]) and is denoted by $S_f(\varepsilon|a)$ or, where no ambiguity is possible, by $S(\varepsilon|a)$.

The condition 4° can be then expressed by saying that the spheres in a semi-pseudo-metric f in R are open in R. This gives together with 1°

(2. A) *Every sphere about a in the semi-pseudo-metric f in R is an open neighbourhood of a in R.*

From (2. A), 2°, 3° is readily deduced

(2. B) *A pseudo-metric f in R is continuous in both variables together* [2, 50].

That is, given a, b in R and $\eta > 0$, there are open neighbourhoods $U(a)$, $U(b)$ in R of a, b respectively such that if $x \in U(a)$ and $y \in U(b)$, then $|f(a, b) - f(x, y)| < \eta$.

If $S(\varepsilon|a)$, $S(\eta|b)$ are two spheres in the semi-pseudo-metric f in R, then we say that $S(\eta|b)$ is *contained* in $S(\varepsilon|a)$ if and only if $f(a, b) < \varepsilon$ and $\eta < \varepsilon - f(a, b)$ (cf. § 3). For a semi-pseudo-metric f this does not imply in general that $S(\eta|b)$ is a subset of $S(\varepsilon|a)$; if, however, f is a pseudo-metric, then from the triangle axiom follows that $S(\eta|b) \subset S(\varepsilon|a)$.

As is well known a property Π of subsets of a T-space R is called *hereditary* (*cogredient* in the nomenclature of HAUSDORFF, cf. [1, 34]) provided that: if a subset A of R has the property Π, every subset B of A also has the property Π.

A property Π of substes of R shall be termed *spherically hereditary with respect to the semi-pseudo-metric f in R* if and only if (2. α) *if a sphere $S_f(\varepsilon|a)$ has the property Π, then every $S_f(\eta|b)$ contained in $S_f(\varepsilon|a)$ also has the property Π.*

If f is a *pseudo-metric* in R, then every hereditary property of R is spherically hereditary with respect to f. The converse is not true.

We call a property Π of subsets of R a *local property of R associated with the semi-pseudo-metric f in R* and say that R is *locally Π with respect to f* if and only if (2. β) *for every point x of R there exists a sphere $S_f(\varepsilon|x)$ of centre x which has the property Π.*

If Π satisfies both (2. α) and (2. β), then Π is termed a *local spherically hereditary property of R associated with f.*

3. *Elements.* If f is a semi-pseudo-metric in a T-space R and A a subset of R, then the supremum of $\{f(x, y)|x, y \in A\}$ is termed the *diameter of A in f* and is denoted by $d_f(A)$ or merely $d(A)$. If the diameter of the whole space R in f is *finite*, then f is said *bounded.*

We consider in the present paper only bounded semi-pseudo-metrics. In the sequel "semi-pseudo-metric" means always "bounded semi-pseudo-metric".

Let Π be a local spherically hereditary property of R associated with the semi-pseudo-metric f in R. For a given point a of R consider the set of all $S_f(\varepsilon|a)$ having the property Π and a radius ε not greater than the diameter $d_f(R) = d$ of R, and denote by ϱ the supremum of their radii.

From the boundedness of f follows that $\varrho(\varrho \leqq d)$ is finite, from (2. β) follows $\varrho > 0$. Then $S_f(\varrho|a)$ will be termed a *Π-element of R in f*, or, where no ambiguity is possible, i.e. where a single property Π and a single semi-pseudo-metric f is considered, merely an *element* of R, a its *centre* and ϱ its *radius of validity*.

Two elements of R will be said equal (notation $=$) if and only if they have equal centres. We will show in the following §§ that the set of all distinct elements of R is an analogue of the set S of all distinct power series with a non vanishing radius of convergence considered in P.V.

Let us first make some simple remarks on elements of R.

From the definition of supremum and from (2. α) follows

(3. A) *If $S(\varrho|a)$ is an element of R, then every sphere $S(\varepsilon|a)$ with the same centre a and with a radius $\varepsilon < \varrho$ has the property Π.*

From (3. A) and (2. α) we get the following generalization of (3. A)

(3. B) *If $S(\varrho|a)$ is an element of R, then every sphere $S(\varepsilon|b)$ which is contained in $S(\varrho|a)$ has the property Π.*

Proof. By hypothesis $0 < \varepsilon < \varrho - f(a, b)$ (cf. § 2), and hence we can put $\varrho - f(a, b) - \varepsilon = 2\eta > 0$. Then $f(a, b) < \varrho - \eta$ and $\varepsilon < (\varrho - \eta) - f(a, b)$. Hence the sphere $S(\varepsilon|b)$ is contained in the sphere $S(\varrho - \eta|a)$, which, by (3. A), has the property Π. (2. α) gives then (3. B).

We consider in (3. C), (3. D), (3. E) a pair of elements of R, namely $S(\varrho|a)$, $S(\sigma|b)$, *such that $f(a, b) < \varrho$.*

From (3. B)

(3. C) *The radius of validity of $S(\sigma|b)$ satisfies the inequality $\sigma \geqslant \varrho - f(a\, b)$.*

From (3. C)

(3. D) *The radius of validity of $S(\sigma|b)$ satisfies the inequality $\varrho + f(a, b) \geqslant \sigma$.*

Proof. If $\varrho + f(a, b) < \sigma$, then $f(b, a) = f(a, b) < \sigma$ and (3. C) can be applied to the pair $S(\sigma|b)$, $S(\varrho|a)$. We obtain $\varrho \geqslant \sigma - f(b, a) = \sigma - f(a, b)$, which is in contradiction to $\varrho + f(a, b) < \sigma$.

From (3. C), (3. D)

(3. E) *The absolute value of the difference of σ and ϱ satisfies the inequality $|\sigma - \varrho| \leqslant f(a, b)$.*

From (3. E) and (2. A) follows

(3. F) *The radius of validity of elements of R considered as function on R is continuous on R.*

Already the above considerations show that elements of R have properties very similar to that of power series elements of an analytic function. The analogy will become even more striking in the subsequent §§.

4. *a direct continuations.* If $S(\varrho|a), S(\sigma|b)$ is an ordered pair of elements of R, then the quotient $A = \frac{f(a, b)}{\varrho}$ will be termed the *relative distance* of the ordered pair $S(\varrho|a)$, $S(\sigma|b)$ and denoted by $S(\varrho|a) - S(\sigma|b) = A$ or by $S(\varrho|a) \overset{A}{-} S(\sigma|b)$. If $A \leqslant \alpha$, then we write $S(\varrho|a) - S(\sigma|b) \leqslant \alpha$ or $S(\varrho|a) \overset{\leqslant \alpha}{-}$

$-S(\sigma|b)$; if $A < \alpha$, $A > \alpha$, $A \geqslant \alpha$, then we replace in the previous formulas the sign $\leqslant$ by $<$, $>$, $\geqslant$ respectively.

$S(\sigma|b)$ will be called *direct continuation* of $S(\varrho|a)$ and written $S(\varrho|a) \rightarrow \rightarrow S(\sigma|b)$ if the relative distance of $S(\varrho|a)$, $S(\sigma|b)$ is less than 1 (i.e. $S(\varrho|a) - S(\sigma|b) < 1$). The formulas $S(\varrho|a) \rightarrow S(\sigma|b) = A$, $S(\varrho|a) \xrightarrow{A} S(\sigma|b)$ etc. have then a sense analogous to the above, the sign — being replaced by $\rightarrow$.

$S(\sigma|b)$ will be said *a direct continuation* of $S(\varrho|a)$ if both $0 < \alpha \leqslant 1$ and $S(\varrho|a) \rightarrow S(\sigma|b) < \alpha$. Remark that in this definition the sign of equality is not valid, i.e. we have really $f(a, b) < \alpha\,\varrho$, but not $f(a, b) = \alpha\,\varrho$. The direct continuations defined before are, in this new nomenclature, 1 direct continuations. Remark that if $S(\sigma|b)$ is α direct continuation of $S(\varrho|a)$ and if $0 < \alpha \leqslant \beta \leqslant 1$, then $S(\sigma|b)$ is also β direct continuation of $S(\varrho|a)$.

The configuration $S(\varrho|a) \rightarrow S(\sigma|b)$ will be termed a *2-chain*. We consider more exactly the 2-chain

(4.1) $$S(\varrho|a) \xrightarrow{A} S(\sigma|b).$$

From (3. C), (3. D)

(4. A) *The radius of validity σ of $S(\sigma|b)$ satisfies the inequality* $(1 + A)\varrho \geqslant \geqslant \sigma \geqslant (1 - A)\varrho$.

From (4. A) and from the symmetry axiom

(4. B) $$S(\sigma|b) - S(\varrho|a) \leqslant \frac{A}{1-A}.$$

Proof. $f(b,a) = f(a,b) = \dfrac{f(a,b)\,(1-A)\,\varrho}{(1-A)\,\varrho} = \dfrac{A}{1-A}(1-A)\,\varrho \leqslant \dfrac{A}{1-A}\,\sigma.$

In exactly the same way we prove

(4. C) $$S(\sigma|b) - S(\varrho|a) \geqslant \frac{A}{1+A}.$$

From (4. B) follows that a sufficient condition for $S(\varrho|a)$ to be direct continuation of $S(\sigma|b)$ is $\dfrac{A}{1-A} < 1$, i.e. $A < \frac{1}{2}$. Thus

(4. D) *If $S(\sigma|b)$ is a direct continuation of $S(\varrho|a)$ and if $\alpha \leqslant \frac{1}{2}$, then $S(\varrho|a)$ is $\left(\dfrac{\alpha}{1-\alpha}\right)$ direct continuation of $S(\sigma|b)$.*

5. *Simple configurations.* In §§ 5, 6 we assume that f is a pseudo-metric. The results of §§ 5, 6 are in general not valid for semi-pseudo-metrics. A consequence is that our theory can be fully developed only for a local spherically hereditary property Π associated with a pseudo-metric f in R. We have considered semi-pseudo-metrics because they are far more general than pseudo-metrics and because all the results of the present Part I with the sole exception of those of §§ 5, 6 and of Theorem (12. B) are valid also for them.

We consider first the configuration

(5. 1) $$S(\tau|c) \xleftarrow{B} S(\varrho|a) \xrightarrow{A} S(\sigma|b).$$

From (4. A) and from the triangle axiom

$$S(\tau|c) - S(\sigma|b) \leqslant \frac{A+B}{1-B}. \tag{5. A}$$

Proof. By (4. A) we have $\tau \geqslant (1-B)\varrho$. From $f(a, b) = A\varrho$, $f(a, c) = B\varrho$, from the symmetry axiom and from the triangle axiom we get $f(c, b) \leqslant (A+B)\varrho$, and hence

$$f(c,b) \leqslant \left(\frac{A+B}{1-B}\right)(1-B)\,\varrho \leqslant \left(\frac{A+B}{1-B}\right)\tau.$$

From (5. A)

(5. B) *To every pair α, α' with $0 < \alpha < \alpha' \leqslant 1$ it is possible to determine β with $0 < \beta < 1$ such that if $S(\sigma|b)$, $S(\tau|c)$ are respectively α, β direct continuations of $S(\varrho|a)$, then $S(\sigma|b)$ is α' direct continuation of $S(\tau|c)$.*

Proof. In the configuration

$$S(\tau|c) \xleftarrow{\beta>} S(\varrho|a) \xrightarrow{<\alpha} S(\sigma|b). \tag{5. 1'}$$

where $0 < \alpha < \alpha' \leqslant 1$ we have to consider $S(\varrho|a) \rightarrow S(\sigma|b) = A < \alpha$ as fixed and to determine β such that $S(\sigma|b)$ be α' direct continuation of $S(\tau|c)$. It suffices to take $\beta = \frac{\alpha'-\alpha}{1+\alpha'}$, for then, by (5. A), we get $S(\tau|c) - S(\sigma|b) < \frac{\alpha+\beta}{1-\beta} = \alpha'$.

Second we consider the configuration

$$S(\varrho|a) \xrightarrow{A} S(\sigma|b) \xrightarrow{C} S(\omega|d). \tag{5. 2}$$

$$S(\varrho|a) - S(\omega|d) \leqslant A + C(1+A). \tag{5. C}$$

Proof. From the triangle axiom and from (4. A) follows

$$f(a, d) \leqslant A\varrho + C\sigma \leqslant A\varrho + C(1+A)\varrho.$$

(5. D) *To every pair α, α' with $0 < \alpha < \alpha' \leqslant 1$ it is possible to determine γ with $0 < \gamma < 1$ such that if $S(\sigma|b)$ is a direct continuation of $S(\varrho|a)$ and $S(\omega|d)$ is γ direct continuation of $S(\sigma|b)$, then $S(\omega|d)$ is α' direct continuation of $S(\varrho|a)$.*

Proof. In the configuration

$$S(\varrho|a) \xrightarrow{<\alpha} S(\sigma|b) \xrightarrow{<\gamma} S(\omega|d) \tag{5. 2'}$$

where $0 < \alpha < \alpha' \leqslant 1$, we have to consider $S(\varrho|a) \rightarrow S(\sigma|b) = A < \alpha$ as fixed and to determine γ such that $S(\omega|d)$ be α' direct continuation of $S(\varrho|a)$. We can take $\gamma = \frac{\alpha'-\alpha}{1+\alpha}$, for by (5. C) we get

$$S(\varrho|a) - S(\omega|d) < \alpha + \gamma(1+\alpha) = \alpha'.$$

6. *Some other configuration-invariants.*

(5. B), (5. D) allow to overlook the configuration

(6. 1) $$S(\tau|c) \overset{\beta>}{\longleftarrow} S(\varrho|a) \overset{<\alpha}{\longrightarrow} S(\sigma|b) \overset{<\gamma}{\longrightarrow} S(\omega|d).$$

(6. A) *To every pair* α, α' *with* $0 < \alpha < \alpha' \leqslant 1$ *it is possible to determine* β, γ *with* $0 < \beta$, $\gamma < 1$ *such that if, in* (6. 1), $S(\tau|c)$ *is* β *direct continuation of* $S(\varrho|a)$ *and* $S(\omega|d)$ *is* γ *direct continuation of* $S(\sigma|b)$, *then* $S(\omega|d)$ *is* α' *direct continuation of* $S(\tau|c)$.

Proof. Choose α'' such that $\alpha < \alpha'' < \alpha'$. Determine, by (5. D), $\gamma(0 < \gamma < 1)$ such that

$$S(\varrho|a) \rightarrow S(\omega|d) < \alpha'', \text{ i.e. take } \gamma = \frac{\alpha''-\alpha}{1+\alpha}.$$

Consider the configuration $S(\tau|c) \overset{\beta>}{\longleftarrow} S(\varrho|a) \overset{<\alpha''}{\longrightarrow} S(\omega|d)$ and determine, by (5. B, $\beta(0 < \beta < 1)$ such that

$$S(\tau|c) \rightarrow S(\omega|d) < \alpha', \text{ i.e. take } \beta = \frac{\alpha'-\alpha''}{1+\alpha'}.$$

We are now able to prove

(6. B) *To every pair* α, α' *with* $0 < \alpha < \alpha' \leqslant 1$ *it is possible to determine* β *with* $0 < \beta < 1$ *such that if* $S(\varrho|a)$ $S(\sigma|b)$ *are* α *direct continuations of one another and* $S(\tau|c)$, $S(\omega|d)$ *are* β *direct continuations of* $S(\varrho|a)$, $S(\sigma|b)$ *respectively, then* $S(\tau|c)$, $S(\omega|d)$ *are* α' *direct continuations of one another.*

Proof. (6. B) is an immediate consequence of the application of (6. A) to the configuration

(6. 2) $$S(\tau|c) \overset{\beta>}{\longleftarrow} S(\varrho|a) \underset{\alpha>}{\overset{<\alpha}{\rightleftarrows}} S(\sigma|b) \overset{<\beta}{\longrightarrow} S(\omega|d).$$

From the proof of (6. A) follows that it suffices to take for β the less of the numbers

$$\frac{\alpha''-\alpha}{1+\alpha}, \ \frac{\alpha'-\alpha''}{1+\alpha'}.$$

The real numbers connected with the configurations considered in §§ 4, 5, 6 never depend on the particular elements of the configuration. For this reason every such number will be called an *element-invariant* or a *configuration-invariant.*

7. *Derived sets. Element-symmetric properties.* If $S(\varrho|a)$ is an element of R, the set of all distinct elements of R which are α direct continuations of $S(\varrho|a)$ will be called the *first* α *derived set* of $S(\varrho|a)$ and will be denoted, as in P.V., by ${}_{\alpha}\{S(\varrho|a)\}$. The *n-th* α *derived set* of $S(\varrho|a)$ is defined by induction and is denoted by ${}_{\alpha}\{S(\varrho|a)\}^{n}$ $(n = 2, 3, \ldots)$. The *infinite* α *derived set* of $S(\varrho|a)$ is the union of *all* finite n-th α derived sets of $S(\varrho|a)$, $n = 1, 2, \ldots$, and is denoted by ${}_{\alpha}\{S(\varrho|a)\}^{\infty}$. Derived sets are infinite con-

figurations, while those considered in § 4, 5, 6 were finite configurations.

Obviously $\alpha \leqslant \beta$ implies

$$(7.1) \qquad {}_{\alpha}\{S(\varrho|a\} \subset {}_{\beta}\{S(\varrho|a)\},$$

and hence, by complete induction

$$(7.2) \qquad {}_{\alpha}\{S(\varrho|a)\}^{n} \subset {}_{\beta}\{S(\varrho|a)\}^{n},\ n = 2,3,\ldots,\ {}_{\alpha}\{S(\varrho|a)\}^{\infty} \subset {}_{\beta}\{S(\varrho|a)\}^{\infty}.$$

In § 8 we shall introduce certain infinite-configuration-invariants, namely certain *derived-set-invariants*, which will play a fundamental rôle in our theory. To this effect we first give in the present § the following definition.

A property Φ of ordered pairs of elements of R will be termed *element-symmetric* if and only if

(7. α) (*Identity*). *If* $S(\varrho|a) = S(\sigma|b)$, *then the pair* $S(\varrho|a)$, $S(\sigma|b)$ *has the property* Φ.

(7. β) (*Symmetry*). *If the pair* $S(\varrho|a)$, $S(\sigma|b)$ *has the property* Φ, *then the "inverse" pair* $S(\sigma|b)$, $S(\varrho|a)$ *has also the property* Φ.

Here are two examples of element-symmetric properties: the property of (7. a) being t direct continuations of one another, t fixed (i.e. $S(\varrho|a)$, $S(\sigma|b)$ has the property Φ if and only if $S(\varrho|a) \underset{t>}{\overset{<t}{\rightleftarrows}} S(\sigma|b)$, $0 < t \leqslant 1$); (7. b) having intersecting (= non disjoint) n-th t derived sets, n, t fixed.

The property (7. a) has a great importance for our theory and will be called the *t direct continuation property* or simply the *t property*. Of course the noun "t property" in reality denotes an uncountable infinity of distinct element-symmetric properties, namely one for each fixed t of the uncountable set $0 < t \leqslant 1$.

8. *Basic integers*. If Φ is an element-symmetric property, then we say that an integer $m \geqslant 2$ is a *Φ basic integer* (where no ambiguity is possible merely *basic integer*) if and only if

(8. α) *there exists a positive integer n and a real number* $\alpha (0 < \alpha \leqslant 1)$ *such that*

(8. β) *for* e v e r y *element* $S(\varrho|a)$ *of R in* e v e r y *sequence of m elements of* ${}_{\alpha}\{S(\varrho|a)\}^{n}$, *say* $S(\varrho_1|a_1)$, $S(\varrho_2|a_2)$, $\ldots$, $S(\varrho_m|a_m)$, *there is a pair of elements with distinct indices, say*

$$S(\varrho_{\mu}|a_{\mu}),\ S(\varrho_{\nu}|a_{\nu}),\ \mu \neq \nu\ (\mu, \nu = 1, \ldots, m),$$

which has the property Φ.

The integer m is said then to *correspond* to n, α, Φ and is denoted by $m(n, \alpha, \Phi)$. An integer m corresponding to n, α, Φ is an invariant (in the sense of § 6) of the n-th α derived set.

In the above definition the elements of the sequence $S(\varrho_1|a_1), \ldots, S(\varrho_m|a_m)$ in (8. β) need not be all distinct. From the identity property (7. α) of Φ follows, however. that m corresponds to n, α, Φ if and only if, for every

$S(\varrho|a)$ of R, in every set of m *distinct* elements of ${}_a\{S(\varrho|a)\}^n$ there is a pair of *distinct* elements with the property Φ.

If Φ is the t property, then the integer $m = m(n, a, \Phi)$ is said, as in P.V., to correspond to n, a, t and is denoted by $m(n, a, t)$.

Obviously

(8. A). *If m corresponds to n, a, Φ, then every integer m' greater than m also corresponds to n, a, Φ.*

From (7. 2)

(8. B) *If m corresponds to n, a, Φ, and if $n_1 \leqslant n$, $a_1 \leqslant a$, then m corresponds also to n_1, a_1, Φ (of course $n_1 =$ positive integer, $a_1 > 0$).*

For the t property we have even the stronger form of (8. B)

(8. B′) *If m corresponds to n, a, t, and if $n_1 \leqslant n$, $a_1 \leqslant a$, $t_1 \geqslant t$, then m corresponds also to n_1, a_1, t_1.*

9. *Bases.* The chief importance of the Φ basic integers is that they lead to the existence of what we call *Φ bases.*

Let W be a set of distinct elements of R and Φ an element-symmetric property. We say that the element $S(\varrho|a)$ of R *has the property Φ with respect to the set W* if and only if

(9. α) *there exists an element $S(\sigma|b)$ of W such that the pair $S(\varrho|a)$, $S(\sigma|b)$ has the property Φ.*

If U is a set of distinct elements of R, then a subset V of U will be called a *Φ basis* of U if and only if

(9. β) *V consists of distinct elements;*

(9. γ) *no pair of distinct elements of V has the property Φ;*

(9. δ) *every element of U has the property Φ with respect to V.*

If Φ is the t property, then we say, as in P.V., simply *t basis.*

We now prove

(9. A) *Theorem. If m corresponds to n, a, Φ, then for every element $S(\varrho|a)$ of R there exists a Φ basis of ${}_a\{S(\varrho|a)\}^n$ consisting of less than m distinct elements.*

Proof. Let $S(\varrho_1|a_1)$ be an arbitrary element of ${}_a\{S(\varrho|a)\}^n$. Then either (1) every element of ${}_a\{S(\varrho|a)\}^n$ has the property Φ with respect to $S(\varrho_1|a_1)$ or (2) there exists an element $S(\varrho_2|a_2)$ of ${}_a\{S(\varrho|a)\}^n$ which has not the property Φ with respect to $S(\varrho_1|a_1)$; in this last case $S(\varrho_2|a_2)$ is distinct from $S(\varrho_1|a_1)$ (identity property (7. α) of Φ!) and the set $S(\varrho_1|a_1)$, $S(\varrho_2|a_2)$ satisfies (9. γ) (symmetry property (7. β) of Φ!). If (1) is valid, $S(\varrho_1|a_1)$ constitutes already a Φ basis of ${}_a\{S(\varrho|a)\}^n$. If (2) is valid, then again either (1. 2) every element of ${}_a\{S(\varrho|a)\}^n$ has the property Φ with respect to $S(\varrho_1|a_1)$, $S(\varrho_2|a_2)$ or (2. 2) there exists an element $S(\varrho_3|a_3)$ of ${}_a\{S(\varrho|a)\}^n$ distinct from $S(\varrho_1|a_1)$, $S(\varrho_2|a_2)$, such that the set $S(\varrho_1|a_1)$, $S(\varrho_2|a_2)$, $S(\varrho_3|a_3)$ satisfies (9. γ). Thus again either we obtain a Φ basis of ${}_a\{S(\varrho|a)\}^n$ or we continue the operation. Since m corresponds to n, a, Φ, the iteration of this process must end after not more than $m - 1$ steps.

Remark. In the definitions (8. α), (8. β), (9. α), (9. β), (9. γ), (9. δ) the symmetry property (7. β) does not occur. Hence it is possible to define basic integers and bases also for properties of ordered pairs of elements of R which satisfy (7. α) only but not (7. β). An example of such a property is the property of being t direct continuation, t fixed, i.e. $S(\varrho|a)$, $S(\sigma|b)$ has the property Φ if and only if $S(\varrho|a) \overset{\leq t}{\rightarrow} S(\sigma|b)$, $0 < t \leqslant 1$. But then the theorem (9. A) is not more valid, i.e. for such "element-assymmetric" properties the existence of a basic integer does not imply the existence of a finite basis. For this reason we consider in the present paper only element-symmetric properties. Φ denotes in the sequel always an element-symmetric property.

10. *The main problem.* In the present § we show that the following situation is of particular interest.

(10. α) *There is a β such that all the basic integers $m(n, \beta, \Phi)$, $n = 1, 2, \ldots$, exist.*

Every β satisfying (10. α) will be called a *remarkable value* of Φ (of course, to be quite exact, it is necessary to add: *with respect to the local property Π considered*). If a remarkable value of Φ exists, Φ is termed *remarkable* (*with respect to Π*).

It is easy to show

(10. A) *Theorem. If α is a remarkable value of Φ, then, for every element $S(\varrho|a)$ of R, every subset V of ${}_{\alpha}\{S(\varrho|a)\}^{\infty}$ that satisfies the conditions* (9. β), (9. γ) *is countable.*

Proof. Associate with every element $S(\sigma|b)$ of V one and only one positive integer n indicating that $S(\sigma|b)$ belongs to ${}_{\alpha}\{S(\varrho|a)\}^{n}$, ${}_{\alpha}\{S(\varrho|a)\}^{n+1}, \ldots$, but not to ${}_{\alpha}\{S(\varrho|a)\}^{n-1}$, and call it the *order of $S(\sigma|b)$ with respect to $S(\varrho|a)$*. Since $m(n, \alpha, \Phi)$ exists, there is only a finite number of elements of V of order $n (n = 1, 2, \ldots)$, and therefore the elements of V can be enumerated according to their order.

From (10. A) follows that, if α is a remarkable value of Φ, then every Φ basis of ${}_{\alpha}\{S(\varrho|a)\}^{\infty}$ is countable. Another question is whether such a basis always exists.

(10. B) *Theorem. If α is a remarkable value of Φ, then, for every element $S(\varrho|a)$ of R, there exists a (countable) Φ basis of ${}_{\alpha}\{S(\varrho|a)\}^{\infty}$.*

Proof. For an arbitrarily fixed positive integer n construct, by (9. A), a finite Φ basis of ${}_{\alpha}\{S(\varrho|a)\}^{n}$, say $S(\sigma_1|b_1)$, $S(\sigma_2|b_2) \ldots, S(\sigma_k|b_k)$. Consider then the subsequent derived set ${}_{\alpha}\{S(\varrho|a)\}^{n+1}$ and, by iterating the operation of the proof of (9. A), complete the just constructed Φ basis of ${}_{\alpha}\{S(\varrho|a)\}^{n}$ to a finite Φ basis of ${}_{\alpha}\{S(\varrho|a)\}^{n+1}$, say $S(\sigma_1|b_1)$, $S(\sigma_2|b_2), \ldots, S(\sigma_k|b_k)$, $S(\sigma_{k+1}|b_{k+1}), \ldots, S(\sigma_{k+l}|b_{k+l})$, the elements $S(\sigma_{k+1}|b_{k+1}), \ldots, S(\sigma_{k+l}|b_{k+l})$ being of order $n+1$ with respect to $S(\varrho|a)$ (cf. the proof of (10. A)). By continuing infinitely this process obtain a countable set V which obviously satisfies (9. β), (9. γ). Since every element of ${}_{\alpha}\{S(\varrho|a)\}^{\infty}$ belongs to a certain

finite derived set ${}_a\{S(\varrho|a)\}^n$, V satisfies also (9. δ) with $U = {}_a\{S(\varrho|a)\}^\infty$.
(10. C) *The theorems* (10. A), (10. B) *remain valid if we replace in them* ${}_a\{S(\varrho|a)\}^\infty$ *by any one of its subsets.*

The theorems (10. A), (10. B) show that, if a is remarkable value of Φ, then the infinite a derived sets have a particularly simple "structure with respect to Φ". This leads to consider the following problem as a fundamental one for our theory.

(10. β) *The main problem. To find necessary and sufficient conditions for a given element-symmetric property* Φ *to be remarkable* (*with respect to a given* Π).

We will give in § 12 a partial solution of the main problem for the t property, namely two sufficient conditions for the t property to be remarkable. The topological importance of such a solution becomes at once evident if one remarks the following fact which is an immediate consequence of (2. A), (4. D).

(10. D) *For every* $a(0 < a \leqslant 1)$ *and for every element* $S(\varrho|a)$ *of* R, *the union of the centres of all elements* $S(\sigma|b)$ *of* ${}_a\{S(\varrho|a)\}^\infty$ *is both open and closed in* R.

From (10. D) follows that for a *connected* R the union of the centres of all elements of ${}_a\{S(\varrho|a)\}^\infty$ is already the whole space R. If then the t property is remarkable, it is possible, by (10. B), to cover R by a countable number of elements.

The case of a non-connected R is more complicated and shall be discussed in detail in Part II.

The more important of the sufficient conditions of § 12 (namely (12. B)) will be derived from § 6 and from a general reasoning which will be developed in § 11.

11. *The condensation principle.* Suppose that m corresponds to n, a, Φ. Consider an indexed set of m^2 (not necessarily distinct) elements of ${}_a\{S(\varrho|a)\}^n$ arranged as an array of m rows, say

$$(11.\ 1)\qquad S_{k1}, S_{k2}, \ldots, S_{km} \qquad (k = 1, 2, \ldots, m)$$

where S_{kl} is written for $S_{kl}(\varrho_{kl}|a_{kl})$ $(k, l = 1, \ldots, m)$. By (8. β) in every row there is a pair of elements with distinct indices which has the property Φ. Suppose the arrangement is such that in each row the pair

$$(11.\ 2)\qquad S_{k1}, S_{k2} \qquad (k = 1, 2, \ldots, m)$$

has the property Φ. Again a pair of elements with distinct indices of the first column of (11. 2), say S_{11}, S_{21}, has the property Φ. Then both pairs S_{11}, S_{12} and S_{11}, S_{21} have the property Φ. Thus we have proved: in every indexed set of m^2 elements of ${}_a\{S(\varrho|a)\}^n$ there are three elements with distinct indices, say S_0, S_1, S_2, such that the both pairs S_0, S_1 and S_0, S_2 have the property Φ. More generally we say that the elements $S_1, S_2, \ldots, S_l$

can be Φ condensed to the element S_0 if and only if the l pairs S_0, S_1; S_0, S_2; ...; S_0, S_l all have the property Φ, and we prove, by complete induction, in exactly the same way as before.

(11. A) *Theorem. If m corresponds to n, a, Φ, then in every indexed set of m^l elements ($l \geqslant 1$) of ${}_a\{S(\varrho|a)\}^n$ there are $(l+1)$ elements with distinct indices such that l among them can be Φ condensed to the remaining $(l+1)$st.*

The reasoning by which (11. A) was proved and sometimes (11. A) itself shall be called the *condensation principle.*

For the t property the condensation principle has the following important consequence.

(11. B) *If m corresponds to n, a, t and l corresponds to $1, t, u$, then m^l corresponds to n, a, u.*

Proof. By (11. A) in every indexed set of m^l elements of ${}_a\{S(\varrho|a)\}^n$ there are $(l+1)$ elements with distinct indices, say $S_0, S_1, \ldots, S_l$, such that $S_1, S_2, \ldots, S_l$ can be "t condensed" to S_0. Since l corresponds to $1, t, u$, there are in the sequence $S_1, \ldots, S_l$ two elements with distinct indices which are u direct continuations of one another.

Remark. In §§ 7 — 11 the triangle axiom has not been used. Hence all the definitions and results of these §§ are valid also for a local spherically hereditary property Π associated with a *semi*-pseudo-metric in R.

12. *Sufficient conditions for the t property to be remarkable.*

(12. A) *Theorem. If a basic integer $m(2, a, t)$ with $0 < t \leqslant a \leqslant 1$ exists, then all basic integers $m(n, a, t)$, $n = 1, 2, \ldots,$ exist.*

Proof. We proceed by complete induction by assuming that the basic integers $m(1, a, t)$, $m(2, a, t)$, ..., $m(n-1, a, t)$ $(n-1 \geqslant 2)$ all exist and we prove the existence of the "subsequent" basic integer $m(n, a, t)$. Since $m = m(n-1, a, t)$ exists, there exists, by (9. A), a finite t basis of ${}_a\{S(\varrho|a)\}^{n-1}$, say $S(\varrho_1|a_1)$, $S(\varrho_2|a_2)$, ..., $S(\varrho_k|a_k)$, $(k < m)$. Since $t \leqslant a$, we have, by (7. 1)

$${}_a\{S(\varrho|a)\}^{n-1} \subset {}_a\{S(\varrho_1|a_1)\} \cup {}_a\{S(\varrho_2|a_2)\} \cup \ldots \cup {}_a\{S(\varrho_k|a_k)\}$$

and hence

$$\text{(12. 1)} \quad {}_a\{S(\varrho|a)\}^n \subset {}_a\{S(\varrho_1|a_1)\}^2 \cup {}_a\{S(\varrho_2|a_2)\}^2 \cup \ldots \cup {}_a\{S(\varrho_k|a_k)\}^2.$$

From (12. 1) follows that in any indexed set of km (not necessarily distinct) elements of ${}_a\{S(\varrho|a)\}^n$ there are m elements with distinct indices, say $S'_1, S'_2, \ldots, S'_m$, which belong to the second a derived set of a single element of the just constructed t basis, say to ${}_a\{S(\varrho_1|a_1)\}^2$. Since $m = m(n-1, a, t)$ $(n-1 \geqslant 2)$, m corresponds (by (8. B)) also to $2, a, t$, and hence in the sequence $S'_1, S'_2, \ldots, S'_m$ there are two elements with distinct indices which are t direct continuations of one another. Thus km corresponds to n, a, t, and (12. A) is proved.

Corollary. If Φ_t is a given t property (t fixed), then a n. a. s. c. for $a \geqslant t$ to be a remarkable value of Φ_t is the existence of the basic integer m $(2, a, t)$.

Notice that (12. A) is valid also for a Π associated with a semi-pseudometric. The triangle axiom (namely (6. B)) allows to establish the following result which can be considered as an improvement on (12. A).

(12. B) *Theorem. If a basic integer $m(1, a, t)$ with $0 < t < a \leqslant 1$ exists, then it is possible to determine β with $0 < \beta < a$ such that all basic integers $m(n, \beta, t)$, $n = 1, 2, \ldots,$ exist.*

Proof. By (6. B) it is possible to determine to the pair t, a in (12. B) a β with $0 < \beta < a$ such that if the elements S_1, S_2, S_3, S_4 form the configuration

$$S_1 \xleftarrow{\beta >} S_2 \underset{t>}{\overset{<t}{\rightleftarrows}} S_3 \xrightarrow{<\beta} S_4 \tag{12. 2}$$

then S_1, S_4 are a direct continuations of one another. By hypothesis $l = m(1, a, t)$ exists, and hence, by (8. B), $m(1, \beta, t)$ exists also. Therefore we assume that $m(1, \beta, t), \ldots, m(n-1, \beta, t)$ all exist and we prove the existence of $m(n, \beta, t)$. Let $m = m(n-1, \beta, t)$ and consider any sequence of m elements of ${}_\beta\{S(\varrho|a)\}^n$, say $S_1', S_2', \ldots, S_m'$. From the definition of the derived sets follows that there exists a sequence of m elements of ${}_\beta\{S(\varrho|a)\}^{n-1}$, say $S_1'', S_2'', \ldots, S_m''$, such that S_k' is β direct continuation of S_k'' for $k = 1, 2, \ldots, m$. Since m corresponds to $n-1, \beta, t$, there are two elements with distinct indices of this last sequence, say S_1'', S_2'', which are t direct continuations of one another. Then the elements S_1', S_1'', S_2'', S_2' form the configuration (12. 2). Hence S_1', S_2' are a direct continuations of one another. Thus we have proved that m corresponds to n, β, a. Since, by hypothesis, $l = m(1, a, t)$ exists, m^l corresponds, by (11. B), to n, β, t, and (12. B) is proved.

The existence of $m(2, a, t)$ in (12. A) and that of $m(1, a, t)$ in (12. B) (a, t fixed) are properties "in the small" of the T-space R considered, more exactly perhaps "properties in the small of R with respect to Π". The existence of all basic integers $m(n, a, t)$, $n = 1, 2, \ldots$ (a, t fixed) is a property "in the large", whose topological importance becomes particularly evident in the case of a connected R (cf. the end of § 10). We can say that (12. A), (12. B) derive from the above properties "in the small" of R the above property "in the large" of R. But, as will be discussed more in detail in Part II, it will be probably very difficult to find a precise set-theoric topological equivalent for the above properties "in the small", in other words, our theory is of purely metric nature.

Copenhagen, July 18, 1949.

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