# A NOTE ON THE THEORY OF STELLAR DYNAMICS. II

BY

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*Abstract.* — The components of average velocity and the values of $\overline{U^2}$, $\overline{V^2}$, $\overline{UV}$ have been derived for a flat, not rotationally symmetrical stellar system with the mass concentrated in the centre. It is found that in the system considered $\overline{U^2}$, $\overline{V^2}$ and $\overline{UV}$ are of the same order of magnitude.

1. *Introduction.* — In the present note we are going to consider a stellar system which in other respects is similar to the system of the preceding note (1), except that the longitudes of the apcentra ($\vartheta_0$) are not distributed at random. Instead we suppose them for one half of the orbits to be equal to zero, and for the other half to be equal to $\pi$. In this way the most "oval" system possible for a given distribution of excentricities is obtained. In order to simplify the mathematical treatment we divide the whole system into two sub-systems, one with $\vartheta_0 = 0$, the other with $\vartheta_0 = \pi$. First we consider the two systems separately and then combine them in order to obtain the final results.

2. *The Motion of Individual Stars.* — The motion of an individual star in the system is described by the same equations as in the previous note [equations (2, 11), (2, 12) and (2, 16)]. For the system with $\vartheta_0 = 0$, we have

$$(2,1)\quad \left\{\begin{aligned} T &= \sqrt{\frac{\mu}{r}}\cdot\sqrt{1-(1-s^2)\cos\vartheta} = \sqrt{\frac{\mu}{r}}\,s\,\sqrt{1+\frac{2(1-s^2)}{s^2}\sin^2\tfrac{1}{2}\vartheta} \\ R &= -\sqrt{\frac{\mu}{r}}\,\frac{(1-s^2)\sin\vartheta}{\sqrt{1-(1-s^2)\cos\vartheta}} = -\sqrt{\frac{\mu}{r}}\,\frac{1-s^2}{s}\,\frac{\sin\vartheta}{\sqrt{1+\frac{2(1-s^2)}{s^2}\sin^2\tfrac{1}{2}\vartheta}} \end{aligned}\right.$$

while for the system with $\vartheta_0 = \pi$

$$(2,1)'\quad \left\{\begin{aligned} T &= \sqrt{\frac{\mu}{r}}\cdot\sqrt{1+(1-s^2)\cos\vartheta} = \sqrt{\frac{\mu}{r}}\,s\,\sqrt{1+\frac{2(1-s^2)}{s^2}\cos^2\tfrac{1}{2}\vartheta} \\ R &= \sqrt{\frac{\mu}{r}}\,\frac{(1-s^2)\sin\vartheta}{\sqrt{1+(1-s^2)\cos\vartheta}} = \sqrt{\frac{\mu}{r}}\,\frac{1-s^2}{s}\,\frac{\sin\vartheta}{\sqrt{1+\frac{2(1-s^2)}{s^2}\cos^2\tfrac{1}{2}\vartheta}} \end{aligned}\right.$$

In both systems, neglecting third order terms of $1 - s^2$,

(2, 2) $$D = -r/s$$

3. *Density and Components of Mean Velocity.* — Let

(3, 1) $$\tfrac{1}{2} f(r_0, T_0, t_0)\, dr_0\, dT_0\, dt_0$$

denote the number of stars with apcentric distances between $r_0$ and $r_0 + dr_0$, apcentric velocities between $T_0$ and $T_0 + dT_0$, and the times of passage through the apcentron between $t_0$ and $t_0 + dt_0$, in the two systems, $\vartheta_0 = 0$ and $\vartheta_0 = \pi$, separately. The integral of the expression (3, 1) over all values of $r_0$, $T_0$ and $t_0$ gives the total number of stars in each of the two systems separately.

In order to obtain the density per unit area, we change the variables

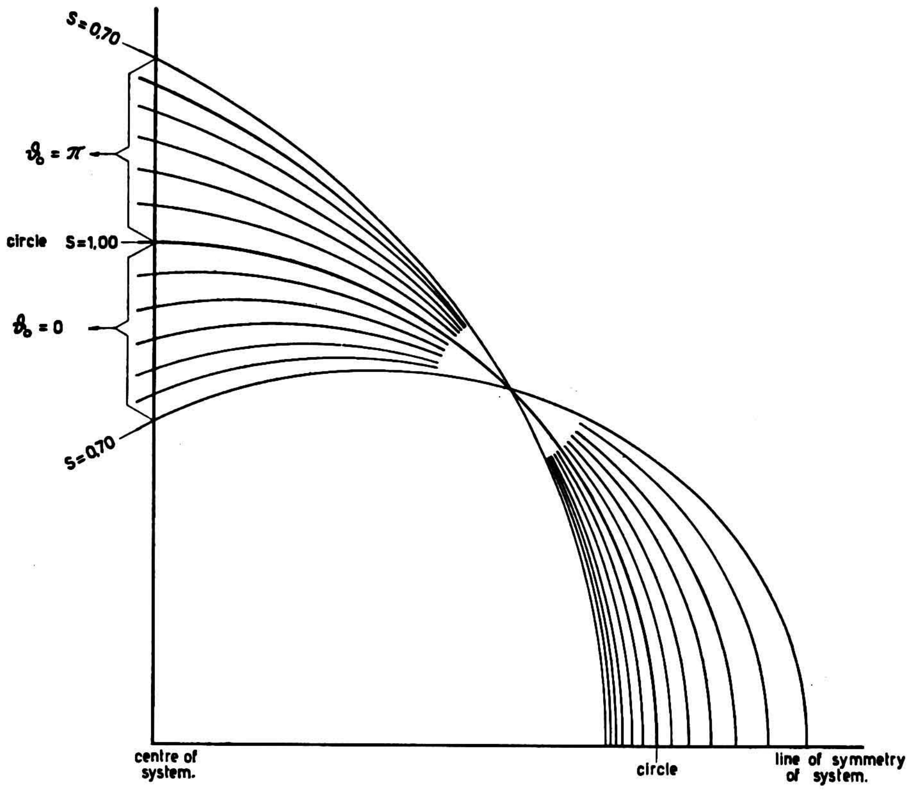


Fig. 1. The figure illustrates the ellipses going through one point, in the system treated in the second note ($s = 0.70, 0.75, \ldots, 0.95, 1.00$; $\vartheta = \pi/4$). The greater axis of each ellipse coincides with the line of symmetry of the system.

$r_0$, $T_0$, $t_0$ in (3, 1) into $r$, $\vartheta$, $s$, then divide the expression by $dr \cdot r d\vartheta$, and finally integrate over $s$. First we have

$$dr_0\, dT_0\, dt_0 = -D \cdot dr\, d\vartheta\, ds.$$

It will be seen that in the present case again the assumption

$$\left\{\begin{array}{ll} f(r_0, T_0, t_0) = N, & \text{from } s = 1 \text{ to } s = s_0 \\ f(r_0, T_0, t_0) = 0, & \text{for } s < s_0, \end{array}\right. \tag{3, 2}$$

where $N = $ const. and $1 - s_0 \ll 1$, leads to a constant density, in the limits of accuracy. We have, for each separate system,

$$\varrho = \tfrac{1}{2} N \int_{s_0}^{1} (1/s)\, ds \tag{3, 3}$$

i.e.

$$\varrho = \tfrac{1}{2} N \ln (1/s_0).$$

In the complete system obtained by superposing the two systems with $\vartheta_0 = 0$ and $\vartheta_0 = \pi$ we have accordingly

$$\varrho = N \ln (1/s_0), \tag{3, 4}$$

or introducing $1 - s = \sigma$; $1 - s_0 = \sigma_0$, which are small quantities,

$$\varrho = N \sigma_0 (1 + \tfrac{1}{2} \sigma_0 + \tfrac{1}{3} \sigma_0^2 + \ldots) \tag{3, 5}$$

The terms written down in this expression are not influenced by the terms neglected in equation (2, 2). We see accordingly that the density is practically independent of $r$ and $\vartheta$, when the function $f(r_0, T_0, t_0)$ is defined by (3, 2).

The next task is to calculate $\overline{T}$ and $\overline{R}$. We have

$$\left\{\begin{array}{l} \varrho \overline{T} = \tfrac{1}{2} N \sqrt{\dfrac{\mu}{r}} \displaystyle\int_{s_0}^{1} \sqrt{1 + \frac{2(1-s^2)}{s^2} \sin^2 \tfrac{1}{2} \vartheta}\, ds \\ \varrho \overline{R} = -\tfrac{1}{2} N \sqrt{\dfrac{\mu}{r}} \sin \vartheta \displaystyle\int_{s_0}^{1} \frac{\dfrac{1-s^2}{s^2}}{\sqrt{1 + \dfrac{2(1-s^2)}{s^2} \sin^2 \tfrac{1}{2} \vartheta}}\, ds \end{array}\right. \tag{3, 6}$$

for the system with $\vartheta_0 = 0$, and

$$\left\{\begin{array}{l} \varrho \overline{T} = \tfrac{1}{2} N \sqrt{\dfrac{\mu}{r}} \displaystyle\int_{s_0}^{1} \sqrt{1 + \frac{2(1-s^2)}{s^2} \cos^2 \tfrac{1}{2} \vartheta}\, ds \\ \varrho \overline{R} = \tfrac{1}{2} N \sqrt{\dfrac{\mu}{r}} \sin \vartheta \displaystyle\int_{s_0}^{1} \frac{\dfrac{1-s^2}{s^2}}{\sqrt{1 + \dfrac{2(1-s^2)}{s^2} \cos^2 \tfrac{1}{2} \vartheta}}\, ds \end{array}\right. \tag{3, 6'}$$

for the one with $\vartheta_0 = \pi$. The expressions can also be written in the form

$$\varrho \overline{T} = \tfrac{1}{2} N \sqrt{\frac{\mu}{r}} \int\limits_{s_0}^{1} \left[1 + \frac{1-s^2}{s^2} \sin^2 \tfrac{1}{2} \vartheta - \tfrac{1}{2} \frac{(1-s^2)^2}{s^4} \sin^4 \tfrac{1}{2} \vartheta + \ldots\right] ds$$

$$\varrho \overline{R} = -\tfrac{1}{2} N \sqrt{\frac{\mu}{r}} \sin \vartheta \int\limits_{s_0}^{1} \left[\frac{1-s^2}{s^2} - \frac{(1-s^2)^2}{s^4} \sin^2 \tfrac{1}{2} \vartheta + \ldots\right] ds$$

for the former system, and

$$\varrho \overline{T} = \tfrac{1}{2} N \sqrt{\frac{\mu}{r}} \int\limits_{s_0}^{1} \left[1 + \frac{1-s^2}{s^2} \cos^2 \tfrac{1}{2} \vartheta - \tfrac{1}{2} \frac{(1-s^2)^2}{s^4} \cos^4 \tfrac{1}{2} \vartheta + \ldots\right] ds$$

$$\varrho \overline{R} = -\tfrac{1}{2} N \sqrt{\frac{\mu}{r}} \sin \vartheta \int\limits_{s_0}^{1} \left[-\frac{1-s^2}{s^2} + \frac{(1-s^2)^2}{s^4} \cos^2 \tfrac{1}{2} \vartheta + \ldots\right] ds$$

for the latter. For the system as a whole we obtain after some calculation:

$$\varrho \overline{T} = N \sqrt{\frac{\mu}{r}} \int\limits_{s_0}^{1} \left[1 + \tfrac{1}{2} \frac{1-s^2}{s^2} - \tfrac{1}{8} \frac{(1-s^2)^2}{s^4} (1 + \cos^2 \vartheta) + \ldots\right] ds$$

$$\varrho \overline{R} = -N \sqrt{\frac{\mu}{r}} \sin \vartheta \int\limits_{s_0}^{1} \left[\tfrac{1}{2} \frac{(1-s^2)^2}{s^4} \cos \vartheta + \ldots\right] ds.$$

Here we may write

$$\frac{1-s^2}{s^2} = 2(1-s) + 3(1-s)^2 + \ldots\ ; \qquad \frac{(1-s^2)^2}{s^4} = 4(1-s)^2 + \ldots$$

With $1 - s = \sigma$ we then have:

$$\varrho \overline{T} = N \sqrt{\frac{\mu}{r}} \int\limits_{0}^{\sigma_0} [1 + \sigma + (1 - \tfrac{1}{2} \cos^2 \vartheta) \sigma^2 + \ldots] d\sigma$$

$$\varrho \overline{R} = -N \sqrt{\frac{\mu}{r}} \sin 2\vartheta \int\limits_{0}^{\sigma_0} [\sigma^2 + \ldots] d\sigma.$$

Integration gives:

$$(3,7) \qquad \begin{cases} \varrho \overline{T} = N \sqrt{\mu/r}\, [\sigma_0 + \tfrac{1}{2} \sigma_0^2 + \tfrac{1}{3} (1 - \tfrac{1}{2} \cos^2 \vartheta) \sigma_0^3 + \ldots] \\ \varrho \overline{R} = -N \sqrt{\mu/r}\, \tfrac{1}{3} \sin 2\vartheta\, [\sigma_0^3 + \ldots] \end{cases}$$

from which, dividing by (3, 5):

$$(3,8) \qquad \begin{cases} v = \overline{T} = \sqrt{\mu/r}\, [1 - \tfrac{1}{6} \cos^2 \vartheta \cdot \sigma_0^2 + \ldots] \\ u = \overline{R} = -\sqrt{\mu/r}\, \tfrac{1}{3} \sin 2\vartheta \cdot \sigma_0^2 + \ldots \end{cases}$$

Accordingly the deviation of the mean motion from a purely circular motion is of the second order of magnitude as regards to $\sigma_0$.

4. *The Quantities* $\overline{U^2}$, $\overline{V^2}$, $\overline{UV}$. — In order to calculate the quantities $\overline{U^2}$, $\overline{V^2}$ and $\overline{UV}$, the individual velocities may be expressed for the first system in the form

$$T = \sqrt{\frac{\mu}{r}}\, s \left[1 + \frac{1-s^2}{s^2} \sin^2 \tfrac{1}{2} \vartheta + \ldots\right] = \sqrt{\frac{\mu}{r}} \left[1 - (1-s) \cos \vartheta + \ldots\right]$$

$$R = -\sqrt{\frac{\mu}{r}} \sin \vartheta \, \frac{1-s^2}{s} + \ldots$$

and for the second system in the form

$$T = \sqrt{\frac{\mu}{r}}\, s \left[1 + \frac{1-s^2}{s^2} \cos^2 \tfrac{1}{2} \vartheta + \ldots\right] = \sqrt{\frac{\mu}{r}} \left[1 + (1-s) \cos \vartheta + \ldots\right]$$

$$R = \sqrt{\frac{\mu}{r}} \sin \vartheta \, \frac{1-s^2}{s} + \ldots$$

For the first system we accordingly obtain, neglecting higher powers of $1-s$,

$$(4, 1) \quad V = T - \overline{T} = -\sqrt{\mu/r}\,(1-s) \cos \vartheta; \quad U = R - \overline{R} = -2 \sqrt{\mu/r}\,(1-s) \sin \vartheta,$$

and for the second system:

$$(4, 2) \quad V = T - \overline{T} = +\sqrt{\mu/r}\,(1-s) \cos \vartheta; \quad U = R - \overline{R} = +2 \sqrt{\mu/r}\,(1-s) \sin \vartheta.$$

The squares of $U$ and $V$ as well as the product $UV$ are accordingly the same in both systems. We obtain:

$$(4, 3) \quad \begin{cases} \varrho\, \overline{U^2} = \frac{4}{3} N \sigma_0^3 \frac{\mu}{r} \sin^2 \vartheta\,; \quad \varrho\, \overline{V^2} = \frac{1}{3} N \sigma_0^3 \frac{\mu}{r} \cos^2 \vartheta\,; \\ \varrho\, \overline{UV} = \frac{1}{3} N \sigma_0^3 \frac{\mu}{r} \sin 2 \vartheta \end{cases}$$

In order to obtain the values of $\overline{U^2}$, $\overline{V^2}$ and $\overline{UV}$ themselves, these expressions have to be divided by $N\sigma_0$. The following list summarizes the values found in sections 3 and 4, up to the lowest power of $\sigma_0$:

$$(4, 4) \quad \begin{cases} u = -\frac{1}{2} \sqrt{\mu/r}\, \sigma_0^2 \sin 2 \vartheta; \quad v = \sqrt{\mu/r}\,(1 - \frac{1}{6} \sigma_0^2 \cos^2 \vartheta); \\ \overline{U^2} = \frac{4}{3} \frac{\mu}{r} \sigma_0^2 \sin^2 \vartheta; \quad \overline{V^2} = \frac{1}{3} \frac{\mu}{r} \sigma_0^2 \cos^2 \vartheta; \quad \overline{UV} = \frac{1}{3} \frac{\mu}{r} \sigma_0^2 \sin 2 \vartheta \end{cases}$$

5. *Conclusions as Regards to Lindblad's Theory of the Formation of Spiral Arms.* — The hydrodynamical equations of motion have already been written down in the first note, but we repeat them here:

$$(5, 1) \quad \begin{cases} \varrho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \vartheta} - \frac{v^2}{r}\right) + \frac{\partial (\varrho \overline{U^2})}{\partial r} + \frac{1}{r} \frac{\partial (\varrho \overline{UV})}{\partial \vartheta} + \frac{\varrho \overline{U^2}}{r} - \frac{\varrho \overline{V^2}}{r} = \varrho \frac{\partial \Phi}{\partial r} \\ \varrho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \vartheta} + \frac{uv}{r}\right) + \frac{\partial (\varrho \overline{UV})}{\partial r} + \frac{1}{r} \frac{\partial (\varrho \overline{V^2})}{\partial \vartheta} + 2 \frac{\varrho \overline{UV}}{r} = 0 \\ \frac{\partial \varrho}{\partial t} + \frac{\partial (\varrho u)}{\partial r} + \frac{1}{r} \frac{\partial (\varrho v)}{\partial \vartheta} + \frac{\varrho u}{r} = 0. \end{cases}$$

The last equation is, of course, the equation of continuity. These equations are satisfied by the values (4, 4), similarly as they are satisfied by the values (4, 4) of the previous note. Now equations (4, 4) show that by making the system sufficiently "oval", the quantities $\overline{U^2}$, $\overline{V^2}$, $\overline{UV}$ will be of equal order of magnitude. Accordingly $\overline{UV}$ cannot be neglected. Besides, there is a certain ratio between $\overline{U^2}$ and $\overline{V^2}$, so that we are not allowed to choose for instance $\overline{U^2} = \overline{V^2}$. If, however, we should choose $\overline{UV} = 0$ and $\overline{U^2} = \overline{V^2}$, this would be equivalent to introducing extra forces on the right hand side of the equations. The consequence would be that $\partial u/\partial t$ and $\partial v/\partial t$ would generally not remain zero. The mass elements can then be expected to follow spirals around the centre. Starting from a slightly oval system, we apparently could obtain spiral arms.

One might think that this conclusion does not necessarily have relation to LINDBLAD's theory of the formation of spiral arms, because we have considered only the particular case in which the whole mass is concentrated in the centre. Spiral nebulae, according to observation, rotate nearly as rigid bodies (2), and must therefore have a quite different distribution of density. However, we have been able to show that at least in one special case spiral arms can be expected to appear precisely with the assumptions which have been made by LINDBLAD, but without any disturbance in the density distribution.

It should moreover be observed that if we calculate the orbit of a star in the system, whatever is the distribution of the density, we calculate $u + U$ and $v + V$ as a whole. A priori there should be no special conditions for $U$ and $V$ (such as $\overline{UV} = 0$, $\overline{U^2} = \overline{V^2}$) as long there are no collisions.

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## REFERENCES

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