DISTRIBUTION MODULO 1 OF SOME CONTINUOUS FUNCTIONS

BY

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This paper consists of two parts. In part I we study an invariant additive set function of sets of real numbers. As an application of the main theorem we obtain the distribution modulo 1 of functions of a certain class. In part II we prove the non-existence of a distribution modulo 1 of functions of another class.

I. *The invariant finitely additive set function ν.*

The space R of the real numbers is group space of the group G of translations $(x \to x + b)$. A set $\mathfrak{S}$ of subsets of R is called *an invariant class of sets*, if:

(1) $$S_1, S_2 \in \mathfrak{S}, \; S_1 \cap S_2 = 0 \text{ implies } S_1 + S_2 \in \mathfrak{S};$$

(2) $$S \in \mathfrak{S}, \; g \in G \text{ implies } g \cdot S \in \mathfrak{S}.$$

A set function defined on $\mathfrak{S}$ is called an *invariant finitely additive set-function (ν)*, if

(3) $$S_1 \cap S_2 = 0, \; S_1, S_2 \in \mathfrak{S}, \text{ implies } \nu(S_1 + S_2) = \nu(S_1) + \nu(S_2);$$

(4) $$S \in \mathfrak{S}, \; g \in G, \text{ implies } \nu(g \cdot S) = \nu(S).$$

We call ν an invariant semi-distribution on $\mathfrak{S}$, if moreover:

(5) $$S \in \mathfrak{S}, \text{ implies } \nu(S) \geqq 0;$$

(6) $$R \in \mathfrak{S}, \text{ and } \nu(R) = 1.$$

Example 1. Throughout the paper $S(x)$ $(x > 0)$ will denote the Lebesgue measure (if existing) of the intersection of a set of real numbers S and the segment $0 - x$. Let $\mathfrak{S}$ be the class of the following sets: a) the sets that are bounded by a greater number (bounded above); b) the sets for which $\lim_{x \to \infty} S(x)/x$ exists; c) any set which is a sum of a set of a) and a set of b).

Throughout the paper $\nu(S)$ will denote the invariant semi distribution on the class of sets $\mathfrak{S}$, defined by: (3); $\nu(S) = 0$ if S is bounded above; $\nu(S) = \lim_{x \to \infty} S(x)/x$ if this exists. We observe that (3) is not contradicted, (5) and (6) are obviously fullfilled, and so we only need to prove (4): If g is

a translation over a distance $d < 0$, and S is a set mentioned under b), then

$$\nu(g \cdot S) = \lim_{x \to \infty} \frac{(g \cdot S)(x)}{x} = \lim_{x \to \infty} \frac{S(x+d) - S(d)}{x} =$$

$$\lim_{x \to \infty} \frac{S(x+d)}{x} = \lim_{x \to \infty} \frac{S(x)}{x} = \nu(S) \quad \text{q.e.d.}$$

Here, and in the sequel, we restrict to proofs concerning sets mentioned under b); the proofs for the other sets are simple consequences.

Example 2. Let $a > 1$, and let $\mathfrak{S}^a$ be the class of sets for which

$$(\nu^a \overset{\text{def}}{=}) \lim_{x \to \infty} \frac{S(ax) - S(x)}{ax - x} \text{ exists.}$$

$S(ax) - S(x)$ denotes the Lebesgue measure of the intersection of S and the interval $(x - ax)$.

The best way to show that also ν^a is a semi distribution on $\mathfrak{S}^a$, is to prove the

Theorem 1. $\mathfrak{S}^a = \mathfrak{S}$, $\nu^a = \nu$, for any $a > 1$.

First we prove a lemma which we will also need later on.

Lemma. Let S be a subset of R, and $a > 1$. *If* for any $\varepsilon > 0$, $N(\varepsilon)$ exists such that $x > N$ implies

$$(7) \qquad \nu' - \varepsilon < \frac{S(ax) - S(x)}{ax - x} < \nu'' + \varepsilon,$$

then $M(\varepsilon, \nu', \nu'')$ exists, such that $y > M$ implies

$$(8) \qquad \nu' - 2\varepsilon < \frac{S(y)}{y} < \nu'' + 2\varepsilon.$$

Proof: For $x > N$, we conclude from (7)

$$\nu' - \varepsilon < \frac{S(a^{k+1} \cdot x) - S(a^k \cdot x)}{a^{k+1} \cdot x - a^k \cdot x} < \nu'' + \varepsilon, \qquad k = 0, 1, 2, \dots$$

or

$$(\nu' - \varepsilon)(a^{k+1} \cdot x - a^k \cdot x) < S(a^{k+1} \cdot x) - S(a^k \cdot x) < (\nu'' + \varepsilon)(a^{k+1} \cdot x - a^k \cdot x).$$

Summation over $k = 0, 1, 2, \dots, n - 1$, yields

$$(\nu' - \varepsilon)(a^n \cdot x - x) < S(a^n \cdot x) - S(x) < (\nu'' + \varepsilon)(a^n \cdot x - x)$$

or if $y = a^n \cdot x$:

$$\nu' - \varepsilon < \frac{S(y) - S(a^{-n} \cdot y)}{y - a^{-n} \cdot y} < \nu'' + \varepsilon.$$

This holds in particular for any $n > 0$ and $N < a^{-n} \cdot y < a \cdot N^2$. It is now possible to change y continuously from N to infinity, while leaving these conditions fullfilled (by shifting the integer n). Hence it follows because $a^{-n} \cdot y$ and $S(a^{-n} \cdot y)$ are bounded, that for y sufficiently large, say $y > M(\varepsilon, \nu', \nu'')$ (8) holds.

Proof of theorem 1:

a. Suppose $S \in \mathfrak{S}^a$. From the definition in example 2, and the lemma, it follows immediately, that also $S \in \mathfrak{S}$, and $\nu^a(S) = \nu(S)$.

b. Suppose $S \in \mathfrak{S}$, S is a set as mentioned in example 1 under b) $\nu(S) = \nu$.

The function $N(\eta)$ defined for $\eta > 0$ exists, such that if $x_2 > x_1 > N(\eta)$, $\eta > 0$ being fixed, then

$$(\nu - \eta)\, x_i < S(x_i) < (\nu + \eta)\, x_i, \qquad i = 1, 2.$$

or

$$(\nu - \eta)\, x_2 - (\nu + \eta)\, x_1 < S(x_2) - S(x_1) < (\nu + \eta)\, x_2 - (\nu - \eta)\, x_1$$

$$\nu - \frac{x_2 + x_1}{x_2 - x_1}\, \eta < \frac{S(x_2) - S(x_1)}{x_2 - x_1} < \nu + \frac{x_2 + x_1}{x_2 - x_1}\, \eta.$$

Let $x_2 = a \cdot x_1 = a \cdot x$, $a = 1 + \delta > 1$, and $\varepsilon > 0$. Choose

$$\eta < \frac{\delta}{2 + \delta}\, \varepsilon.$$

Then

$$\nu - \varepsilon < \frac{S(ax) - S(x)}{ax - x} < \nu + \varepsilon \qquad (x > N(\eta) = N(\eta(\varepsilon))).$$

This holds for any $\varepsilon > 0$ and $x > N(\eta(\varepsilon))$, hence:

$$\nu^a(S) \stackrel{\text{def}}{=} \lim_{x \to \infty} \frac{S(ax) - S(x)}{ax - x} = \nu$$

and $S \in \mathfrak{S}^a$. The theorem follows.

Example 3. Let the function $y = f(x)$ be defined and be monotonously increasing for $x > x_0$, and $f(x) < f(x_0)$ or not defined for $x < x_0$. Let $\mathfrak{S}^* = f^{-1}(\mathfrak{S})$ consist of the subsets of R that are the sum of a bounded (above) set and the image under f^{-1} of a set of $\mathfrak{S}$ (example 1) (it is clear that choice of a larger number then x_0, instead of x_0, has no influence on the result $\mathfrak{S}^*$. The function $f(x)$ is only of interest for values $x >$ the arbitrarily large value $x_0(!)$).

An additive set function ν^* on $\mathfrak{S}^*$ is defined by (3) and: $\nu^*(S^*) = 0$ if $S^*(\in \mathfrak{S}^*)$ is bounded; if $S^* = f^{-1}(S)$ then

$$\nu^*(S^*) = \nu^*(f^{-1}(S)) = \nu(S) = \nu(f(S^*)).$$

Notation: $\mathfrak{S}^* = f^{-1}(\mathfrak{S})$, $\nu^* = \nu \cdot f$.

The following theorem gives a condition under which the setfunction just defined is an invariant semi distribution, and it is even the same as ν in example 1.

Main Theorem 2:

If $a > 0$, x_0 is a constant, $f(x)$ is bounded above or not defined for $x < x_0$, $f(x)$ is differentiable for $x \geqq x_0$, and

(9) $$\lim_{x \to \infty} \frac{f'(x)}{a \cdot x^{a-1}} = K > 0,$$

then $\mathfrak{S}^* = f^{-1}(\mathfrak{S}) = \mathfrak{S}$ *and* $\nu^* = \nu \cdot f = \nu \cdot$; *or in words : then the semi distribution ν is invariant under the "transformation" f.*

Proof: Let $K = 1$ (a restriction not essential for the proof). From (9) follows the existence of $N'(\varepsilon) > x_0$, defined for $\varepsilon > 0$, such that $x > N'(\varepsilon)$ implies:

$$(10) \qquad a(1-\varepsilon)\, x^{a-1} < f'(x) < a(1+\varepsilon)\, x^{a-1}.$$

We integrate from N' to z and replace z by x:

$$(1-\varepsilon)\, x^a + C_1 < f(x) - f(N') < (1+\varepsilon)\, x^a + C_2.$$

Therefore $N''(\varepsilon) > N'(\varepsilon)$ exists, such that $x > N''(\varepsilon)$ implies (10) and

$$(11) \qquad (1-2\varepsilon)\, x^a < f(x) < (1+2\varepsilon)\, x^a.$$

Now let $\nu^* = \nu^*(S^*)$ be the value of the setfunction $\nu^* = \nu \cdot f$ at the set $S^* = f^{-1}(S)$ $(S = f(S^*))$. Then because $\nu^* = \lim_{x \to \infty} S(f(x))/f(x)$, $N^0(\eta)$ defined for $\eta > 0$ exists, such that $x > N^0(\eta)$, $a > 1$, implies

$$(\nu^* - \eta)\, f(ax) < S(f(ax)) < (\nu^* + \eta)\, f(ax),$$

$$(\nu^* + \eta)\, f(x) > S(f(x)) > (\nu^* - \eta)\, f(x),$$

$$(\nu^* - \eta)\, f(ax) - (\nu^* + \eta)\, f(x) < S(f(ax)) - S(f(x)) < (\nu^* + \eta)\, f(ax) - (\nu^* - \eta)\, f(x),$$

$$(12) \qquad \nu^* - \frac{f(ax) + f(x)}{f(ax) - f(x)}\, \eta < \frac{S(f(ax)) - S(f(x))}{f(ax) - f(x)} < \nu^* + \frac{f(ax) + f(x)}{f(ax) - f(x)}\, \eta.$$

From now on we suppose $a \geq 1$. The proof of the other case $0 < a < 1$ is obtained by obvious alterations.

The derivative of the function f in the interval $x - ax$ $(x > N''(\varepsilon)$ and $x > N^0(\eta))$ is bounded by $a(1-\varepsilon)\, x^{a-1}$ and $a(1+\varepsilon)\, (ax)^{a-1}$ (compare (10)). Therefore $f(ax) - f(x)$ is bounded by (!):

$$(13) \qquad a(1-\varepsilon)\, x^{a-1}\, (ax - x) \quad \text{and} \quad a(1+\varepsilon)\, (ax)^{a-1}\, (ax - x).$$

The Lebesgue measure $S(f(ax)) - S(f(x))$ is bounded by (!):

$$(14) \quad a(1-\varepsilon)\, x^{a-1}\, (S^*(ax) - S^*(x)) \quad \text{and} \quad a(1+\varepsilon)\, (ax)^{a-1}\, (S^*(ax) - S^*(x)).$$

From (13) and (14) we get:

$$(15) \quad \frac{1-\varepsilon}{1+\varepsilon}\, a^{1-a}\, \frac{S^*(ax) - S^*(x)}{ax - x} < \frac{S(f(ax)) - S(f(x))}{f(ax) - f(x)} < \frac{1+\varepsilon}{1-\varepsilon}\, a^{a-1}\, \frac{S^*(ax) - S^*(x)}{ax - x}$$

From (11) and (13) we get:

$$(16) \qquad \frac{f(ax) + f(x)}{f(ax) - f(x)}\, \eta < \frac{(1+2\varepsilon)\, [(ax)^a + x^a]}{a(1-\varepsilon)\, x^{a-1}\, (ax - x)}\, \eta = \frac{1+2\varepsilon}{a(1-\varepsilon)}\, \frac{a^a + 1}{a - 1}\, \eta.$$

We now choose for any $a > 1$, $\varepsilon > 0$, $\eta = \eta(a, \varepsilon)$ so small that the right hand side of (16) is less then ε. This inequality is used in (12) and combining the result with (15) we obtain:

$$(17) \qquad \frac{1-\varepsilon}{1+\varepsilon}\, a^{1-a}\, (\nu^* - \varepsilon) < \frac{S^*(ax) - S^*(x)}{ax - x} < \frac{1+\varepsilon}{1-\varepsilon}\, a^{a-1}\, (\nu^* + \varepsilon).$$

Applying the lemma we conclude to the existence of $M(a, \varepsilon')$, defined for all $a > 1$, $\varepsilon' > 0$, such that $x > M(a, \varepsilon')$, a, ε' being fixed, implies

$$\nu^* a^{1-a} - \varepsilon' < \frac{S^*(x)}{x} < a^{a-1} \nu^* + \varepsilon'$$

hence

$$\lim_{x \to \infty} \frac{S^*(x)}{x} = \nu^*.$$

By definition this limit is $\nu(S^*)$, and therefore $S^* \in \mathfrak{S}$. It follows that $\mathfrak{S}^* \subset \mathfrak{S}$. Because $f(x)$ has an inverse for $x > x_0$, which obeys conditions (9) (with other constants), also $\mathfrak{S} \subset \mathfrak{S}^*$. Therefore $\mathfrak{S}^* = \mathfrak{S}$, and for any $S \in \mathfrak{S} : \nu^*(S) = \nu(S)$. q.e.d.

Application.

Let s be a Lebesgue measurable (L.m.) subset of the interval $0 \le x < 1$, with measure $\mu(s)$. Let $S = S(s)$ be the set of all numbers x that differ an integer from a number in the set s. Obviously $\nu(S) = \mu(s)$.

Let $f(x)$ be a function and let $S^*(s) = f^{-1}(S)$ be the set of all x for which $f(x)$ is a number in the set S. If $\nu(S^*) = \nu(S^*(s), f) = \nu(s, f)$ exists for any L.m. set s, and $\nu(s, f)$ is an infinitely additive setfunction, then $f(x)$ is said to possess the C^{III}-*distribution mod* 1: $\nu(s, f)$. If $\nu(s, f)$ exists for any s that is a finite sum of intervals, and $\nu(s, f)$ is finitely additive, then $f(x)$ is said to possess the C^{I}-*distribution mod* 1: $\nu(s, f)$. If the distribution mod 1 $\nu(s, f)$ coincides with the Lebesgue measure $\mu(s)$, then the distribution is called *uniform*.

As a *corrollary of theorem* 2 we now have:

Theorem 3 (KUIPERS–MEULENBELD). *If $f(x)$ obeys the conditions of theorem* 2 *(in particular* (9)*), then $f(x)$ is C^{III} uniformly distributed mod* 1. [1]

Proof: If s is L.m. then $\nu(s, f) = \nu(S^*) = \nu(S) = \mu(s)$.

II. *Some functions which do not possess a C^{I}-distribution mod* 1.

In theorem 2 we proved the invariance of the finitely additive setfunction $\nu(S)$ under a transformation which is in a certain sense not much different from $x \to x^a$, $a > 0$. It is easily seen that $\nu(S)$ is *not* invariant under $x \to e^x$ or the inverse $x \to \ln x$. For these functions the conclusion of theorem 3 is not a corrollary of theorem 2. e^x happens to be C^{III}-uniformly distributed mod 1; $\ln x$ not. We prove:

Theorem 4. *If $M, L > 0$ are constants, $f(x)$ is continuous, $\lim_{x \to \infty} f(x) = \infty$* [2]*), and if $\gamma > \beta > M$, $f(\gamma) - f(\beta) > 1/4$ implies*

$$(18) \qquad \beta \frac{f(\gamma) - f(\beta)}{\gamma - \beta} < L,$$

then $f(x)$ does not possess a C^{I}-distribution mod 1.

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[1] For $a \ge 1$, this is, but for a condition of monotony which I do not need, a theorem of KUIPERS ([2] Ch. III, th. 6). For $0 < a < 1$, KUIPERS and MEULENBELD recently gave a proof of an n-dimensional generalisation of th. 3 ([4] th. IV).

[2] This condition is superfluous, but it is convenient in the proof.

Proof: Suppose the finitely additive setfunction $\nu(s) = \nu(s, f)$ is the C^I-distribution mod 1 of the given function $f(x)$. We will show that the assumption of its existence leads to a contradiction.

Let s be the interval of numbers x which obey $0 \leqq p < x < q < 1$, $q - p = b > 1/4$. $S = S(s)$ and $S^* = f^{-1}(S)$ are defined as before.

By definition $N(\varepsilon)$ exists, such that $x > N(\varepsilon)$ implies

$$(19) \qquad \nu(s) - \varepsilon < \frac{S^*(x)}{x} < \nu(s) + \varepsilon.$$

Let also $N(\varepsilon) > M$.

Next we choose two numbers β and $\gamma > \beta > N(\varepsilon)$, such that: γ is the smallest number greater than β, for which $f(\gamma) \equiv q$ mod 1; β is the greatest number smaller than γ, for which $f(\beta) \equiv p$ mod 1; $f(\gamma) > f(\beta)$. Because $f(x)$ is continuous:

$$S^*(\gamma) = S^*(\beta) + (\gamma - \beta).$$

In view of (19):

$$(20) \qquad S^*(\gamma) > (\nu(s) - \varepsilon)\beta + (\gamma - \beta) = \gamma - (1 - \nu(s) + \varepsilon)\beta.$$

Application of (18) yields:

$$\frac{\beta b}{\gamma - \beta} < L \text{ or } \beta < \frac{L}{L+b}\gamma.$$

Substitute in (20), divide by γ, and rearrange:

$$(21) \qquad \frac{S^*(\gamma)}{\gamma} > \nu(s) + \frac{(1 - \nu(s))\,b}{L+b} - \varepsilon \frac{L}{L+b}.$$

If $\nu(s) \neq 1$, then ε can be choosen so small, and $\gamma > N(\varepsilon)$ exists, such that

$$\frac{S^*(\gamma)}{\gamma} > \nu(s) + \varepsilon$$

in contradiction with (19).

Hence for all intervals like s we get the same value $\nu(s) = 1$ so that $\gamma(s)$ cannot be an additive setfunction · q.e.d.

Examples:

1. If $f(x)$ is differentiable and $x \cdot f'(x)$ $(x > 0)$ is bounded, then $f(x)$ does not possess a C^I-distribution mod 1. In particular $f(x)$ is not uniformly distributed, which was also proved by KUIPERS and MEULENBELD ([4] th. II). E.g. $\ln x$, $\ln (x + \sin x + 1)$, have no C^I-distribution mod 1.

2. A step function with discontinuities at the integervalues of x, can be approximated by a continuous function in such a way, that the measure of the set of all x for which the values of the two functions differ, is bounded. We are therefore able to prove: If the sequence $n(f(n + 1) - f(n))$ $(n > 0)$ is bounded, then the function $f(x)$, defined by $f(x) = f(n)$

if $n \leqq x < n + 1$, does not possess a C^{I}-distribution mod 1; in other words: $f(n)$ does not possess a C^{I}-distribution mod 1.

Compare [1] E.g. $f(n) = \ln n$.

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