REALISATIONS UNDER CONTINUOUS MAPPINGS

BY

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1. *Introduction and summary*

To what extent is a continuous mapping topological on a well chosen subset?

Given a continuous mapping $f(M) = M'$ of a space M on a space M', one may ask for the existence of a subset $M^* \subset M$ such that

$$f(M^*) = M'$$

and f even *topological* (on M^*). If this is possible, we call M^* a homeomorphic *realisation* of M' in M under f.

Furthermore, f defined on M^* is called a (topological) realisation of f (defined on M). Since we may remark, that in this case

$$(1) \qquad\qquad f^{-1} f(M) = M^*$$

(f^{-1} now denoting the topological inverse mapping of M'), M^* is a retract of M, the retraction given by $f^{-1} f$.

Conversely, any retraction

$$f(M) = A \subset M$$

is determining a trivial realisation A of A.

It will be shown in 4. by some examples, that very strong conditions are required, imposed on M and f, if a realisation will be possible.

But even then I cannot obtain general conclusions.

The mapping $A \times B \to B$ (defined by $A \times b \to b$, $b \in B$) of a topological product on one of its factors, gives a simple example of a mapping by which a realisation is possible.

One might consider realisations M^* in connection with M as a certain generalisation of the topological product concept. This is one reason why realisations seem to me of some interest.

Realisations generally being impossible, one may ask how far realisations are possible with respect to certain subsets of M'. In this way we arrive at the following definition.

Be given a continuous mapping $f(M) = M'$. If it is possible to find a subset $^*M$ of M such that f is topological on $^*M$ and the image $^*M'$ of $^*M$ under f is *dense* in M', we define $^*M$ as a *weak (topological) realisation* of M' in M under f.

Shortly, *M is a weak realisation, if

$$\overline{f(*M)} = M', \qquad f \ topological \ (\text{on } *M).$$

Weak realisations are not always possible. If M denotes the set of a countable number of isolated points and this set is mapped one to one on the set M' of rational numbers, obviously this mapping is continuous but there does not exist any weak realisation. The situation is completely changed however, if we are considering continuous mappings of *compact* sets.

We shall prove (see theorem III and III'), that to *any continuous mapping of an arbitrary compact metric space corresponds a weak realisation.*

We obtain this result by using an important theorem of HILL [8] and KURATOWSKI [7], which essentially says that, given an upper semi-continuous decomposition of a compact metric space, this decomposition is continuous on a well chosen subset, while the corresponding set in the decompositionspace (hyperspace) is dense in this space [1]).

This contention may be interpreted as a weak (interior) realisation in respect to *interior* mappings instead of the formerly mentioned topological mappings (theorem II).

This "weak *interior*-realisation theorem" is established by STOILOW [10]. This theorem is moreover an almost immediate consequence of the theorem of HILL and KURATOWSKI.

Our weak topological realisation is somewhat strengthened in theorem III' by extension of the topological mapping, giving us the following mainresult: *any continuous] mapping f of a compactum M is topological on a G_δ-subset S of M, such that the G_δ-set $f(S)$ is dense in $f(M)$.*

This result intersects with theorems of KURATOWSKI, HUREWICZ and STOILOW, our result being far stronger but only proved in compact spaces. KURATOWSKI [6], p. 227 proved that any continuous mapping of a complete separable space is topological on a certain set D being a discontinuum of CANTOR, provided that the image is an uncountable set.

Now considering a compactum instead of a complete space this fact is a simple consequence of theorem III'. Indeed, the uncountable set $f(M)$ contains a compact subset K dense in itself; according to theorem III' there exists a G_δ-subset S of the compact set $f^{-1}(K)$ such that f is topological on S and $f(S)$ is a G_δ-subset of K dense in K. $f(S)$ is therefore a G_δ-set (therefore topologically complete), dense in itself and contains a discontinuum of CANTOR D' according to a theorem of YOUNG (compare e.g. HAHN [11], p. 127).

The topological inverse of D' gives the required D. HUREWICZ [12] gives generalisations of this theorem of KURATOWSKI; for instance: a

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[1]) KURATOWSKI's theorem is expressed as follows: "La famille des tranches de continuité est un ensemble G_δ dense dans l'hyperespace". For a partial result see also MOORE [9], p. 348, theorem 24.

continuous mapping of a separable space with an uncountable image is topological on a certain perfect set.

Finally: STOILOW [10] proves a theorem of very restricted character assuming that M and M' are n-dimensional manifolds and that the inverse set $f^{-1}(m')$ of any point $m' \in M'$ is countable.

All mentioned spaces are separable metric, all mappings continuous. Commonly we are using the terminology of WHYBURN [1]. The realisation problem originates from a suggestion of J. CH. BOLAND.

2.   *Theorem I.* (HILL, KURATOWSKI). *If $f(M) = M'$ is a continuous mapping of the compact metric space M, the corresponding upper semicontinuous decomposition $\{f^{-1}(x)\}$ $(x \in M')$ of M contains a sub-collection being a continuous decomposition, while the corresponding image under f of this sub-collection in M' (M' is homeomorphic with the decomposition-space (hyperspace) of M) is dense in M'.*

2. 1.   We can express theorem I by means of interior mappings. To this end we call $M_* \subset M$ a *weak interior realisation* of M' in M under f, $f(M) = M'$ if

$$\overline{f(M_*)} = M', \qquad f \text{ interior on } M_*.$$

As a simple result of theorem I we obtain

Theorem II. (STOILOW). *At any continuous mapping of a compact metric space a weak interior realisation may be found* [2]).

Proof.   Applying theorem I to the upper semicontinuous decomposition performed by the mapping $f(M) = M'$, we obtain a subcollection $M_* = \Sigma E$, being continuous (in the limit sense), such that

$$\overline{f(M_*)} = M'$$

$M_* = \Sigma E$ is a total inverse set under f. Hence we have only to prove that f is interior on M_* (if M_* would be compact, this fact is expressed by a well known theorem, but M_* is not compact in general).

If a neighbourhood $U(p \mid M_*)$ of a point

$$p \in E \subset M_*$$

is not transformed on a neighbourhood of $p' = f(p)$ in $f(M_*)$, there may be found a sequence of points $p_i' \in f(M_*)$ converging to p' such that the intersection of $f^{-1}(p_i')$ and $U(p|M_*)$ is vacuous. A subsequence of points $\{p_l\}$

$$p_l \in f^{-1}(p_{i_l}') \subset M_*$$

is converging to a point $q \in M$; q is a point of E, since f is continuous, and $f(q) = \lim f(p_l) = f(p) = p'$ gives us

$$q \in f^{-1}(p') = E$$

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[2])   *Note.* We shall prove even more. *In fact M_* is a total inverse set under f:*

$$M_* = f^{-1} f(M_*)$$

Taking this into consideration we see that theorem II is not an immediate consequence of theorem III.

Therefore

$$\lim \inf f^{-1}(p'_{i_l}) \cdot E \neq 0.$$

This leads to (compare note 1))

$$\lim f^{-1}(p'_{i_l}) = E.$$

Applying this result we can find in our neighbourhood $U(p|M_*)$ a point p'_k contrary to the fact that the intersection of $f^{-1}(p'_i)$ and $U(p|M_*)$ is vacuous. The transformed set of $U(p|M_*)$ is therefore a neighbourhood of $f(p)$ in $f(M_*)$; thus is f interior on M_*.

3. *Theorem III. At any continuous mapping of a compact metric space a weak (topological) realisation may be found.*

It is worth noting, that the points of the realisation set $^*M \subset M$ must be carefully chosen.

We are giving a simple example for illustration. Let M be the plane set

$$-1 \leqslant x \leqslant 0 \quad , \qquad y = 1$$
$$x = 0 \quad , \quad 1 \leqslant y \leqslant 2$$
$$0 \leqslant x \leqslant 1 \quad , \qquad y = 2$$

Let f be the projection of M on the X-axis, such that M' is the set

$$-1 \leqslant x \leqslant 1 \quad , \quad y = 0$$

It is clear that there does not exist any (topological) realisation. A weak realisation $^*M$ however is given e.g. by the set

$$-1 \leqslant x < 0 \quad , \quad y = 1$$
$$0 < x \leqslant 1 \quad , \quad y = 2$$

On the other hand it is apparently impossible to find a weak realisation $^*M$ which contains a point of the total inverse set

$$x = 0 \quad , \quad 1 \leqslant y \leqslant 2$$

of

$$x = 0, y = 0.$$

Proof of theorem III. We put as before,

$$f(M) = M'$$

M and M' compact metric with metrics ϱ and ϱ'.

$$M_* = \Sigma E_*$$

is a continuous subcollection of the corresponding upper semi-continuous decomposition $M = \Sigma E$, while $M'_* = f(M_*)$ is dense in $f(M)$. All this is possible according to theorem I. We start changing the given metric ϱ' in M'_*.

To obtain this metric we are using a well known distance function α

of the compact subsets of M. Given two compact subsets C_1 and C_2 of M, we define

$$a(C_1, C_2) = \inf \beta$$

if β is a real number such that

$$U_\beta(C_1) \supset C_2 \text{ and } U_\beta(C_2) \supset C_1.$$

It is known (comp. [5], p. 115) that in a compact space "metric" convergence of compact subsets imposed by this distance function is identical with the "topological" convergence.

Now we contend, that

$$\tilde{\varrho}(x', y') = \varrho'(x', y') + a(f^{-1}(x'), f^{-1}(y')), \qquad x', y' \in M'_*$$

is a metric in M'_* equivalent to the given metric ϱ' (it is worth noting that generally this is not true if we define $\tilde{\varrho}$ on M'!). We see at once that $\tilde{\varrho}$ is a distance function since ϱ' and a are distance functions. Further, if

$$(1) \qquad \lim_i \varrho'(x'_i, x') = 0 \quad, \quad x'_i, x' \in M'_*$$

we have

$$\lim_i f^{-1}(x'_i) = f^{-1}(x'),$$

since the decomposition of M_* is continuous. This convergence being identical with metric convergence, we obtain

$$\lim_i a(f^{-1}(x'_i), f^{-1}(x')) = 0$$

and therefore

$$\lim_i \tilde{\varrho}(x'_i, x') = 0.$$

Conversely this relation obviously implies the relation (1).

M'_* (the imposed metric is from now on $\tilde{\varrho}$) has *finite ε-coverings* with mesh $\leqslant \varepsilon$ for any $\varepsilon > 0$. This will be proved, if any sequence $\{p'_i\}$ of points $p'_i \in M'_*$ has a fundamental subsequence, according to a known theorem (comp. [5], p. 104).

To prove this contention we select from $\{p'_i\}$ a subsequence converging at $q' \in M'$, say $\{q'_k\}$. Thus for any $\varepsilon > 0$ we may find a natural number N_1 such that

$$(2) \qquad \varrho'(q'_l, q'_m) < {}^1/_2\,\varepsilon \text{ if } l, m > N_1.$$

The collection

$$\{f^{-1}(q'_k)\} = \{{}_k E_*\}$$

is a collection of E_*— sets and has a convergent subsequence $\{{}_l E_*\}$, its limit being contained in $f^{-1}(q')$.

Obviously

$$(3) \qquad a({}_{l_1} E_*, {}_{l_2} E_*) < {}^1/_2\,\varepsilon \text{ if } l_1, l_2 > N_2.$$

Now from (2) and (3) we immediately obtain

$$\tilde{\varrho}\,(q'_{l_1}, q'_{l_2}) < \varepsilon \text{ if } l_1, l_2 > N_3.$$

Hence $\{q'_i\}$ is a fundamental subsequence of $\{p'_i\}$.

Let A'_* be a countable subset of M'_* dense in M'_* and therefore dense in M'.

There exists a real number $\gamma_1 < 1$ such that the neighbourhood $U_{1/2 \gamma_1}(p'|A'_*)$ of any point $p' \in A'_*$ is open and closed in A'_*. This results from the countability of A'_* and the non-countability of the set of real numbers between 0 and 1.

Therefore we may determine a finite open and closed γ_1-covering (with mesh $\leqslant \gamma_1$) $\{U_i\}$ ($i = 1, 2, \ldots, k$) of A'_*.

This is possible since $A'_* \subset M'_*$. Putting

$$V_i = U_i - \sum_{j=1}^{i-1} U_j \quad (i = 1, 2, \ldots, k),$$

we obtain a finite γ_1-covering $O_1 = \{V_i\}$ of A'_* of disjoint sets both open and closed in A'_* with mesh $\leqslant \gamma_1$.

Similarly it is apparently possible to determine a refinement of this covering being a finite γ_2-covering $O_2 = \{V_{ij}\}$ consisting of disjoint sets both open and closed in A'_* such that

$$\gamma_2 < {}^1\!/_2\, \gamma_1 \quad , \quad V_{ij} \subset V_i.$$

Continuing this process we obtain a sequence of subsequently refined coverings $\{O_i\}$, each O_i being a finite γ_i-covering

$$\gamma_i < {}^1\!/_2\, \gamma_{i-1}$$

consisting of disjoint sets both open and closed in A'_*. Now we construct a sequence of points $\{a'_i\}$, $a'_i \in A'_*$ as follows; in each V_k of O_1 we select one point a'_k ($k = 1, 2, \ldots, k_1$); in each V_{rj} of O_2 we select one point a'_{k_1+i} ($l = 1, 2, \ldots, l_1$), except for a V_{rj} which already contains a point a'_k. In each V_{rjs} not containing a point a'_k or a_{k_1+l} we select a point

$$a'_{k_1+l_1+m} \qquad (m = 1, 2, \ldots, m_1)$$

and so on ad infinitum.

The countable set

$$A' = \varSigma\, a'_i \subset A'_* \subset M'_* \subset M'$$

is obviously dense in M'.

Now we start transforming A' one to one on a set $A \subset M$, such that

$$f(A) = A'.$$

Beginning with the points a'_k ($k = 1, 2, \ldots, k_1$) we select in each E_*-set $f^{-1}(a'_k) = E^k_*$ an arbitrary chosen point a_k. Proceeding with the points a'_{k_1+l} ($l = 1, 2, \ldots, l_1$), obtained at the covering O_2, we select in each

$$f^{-1}(a'_{k_1+l}) = E^{k_1+l}_*$$

a point a_{k_1+l} such that

(4) $$\varrho\,(a_{k_1+l},\,a_k) = \min\varrho\,(E_*^{k_1+l},\,a_k),$$

k being determined by the conditions

$$a'_{k_1+l} \in V_k,\; f\,(a_k) = a'_k \in V_k \in O_1.$$

In other words: a'_{k_1+l} is contained in one and only one V_k; select the corresponding a'_k in V_k and determine $a_{k_1+l} \in E_*^{k_1+l}$ according to (4).

Continuing this process we select in each

$$f^{-1}\,(a'_{k_1+l_1+m}) = E_*^{k_1+l_1+m}$$

a point $a_{k_1+l_1+m}$ such that

(5) $$\varrho\,(a_{k_1+l_1+m},\,a_\pi) = \min\varrho\,(E_*^{k_1+l_1+m},\,a_\pi),$$

π fixed, being exactly one of the numbers $1,\,2,\,\ldots,\,k_1+l_1$, determined by the conditions

$$a'_{k_1+l_1+m} \in V_{rj},\; a'_\pi \in V_{rj} \in O_2.$$

Indefinitely continuing this process defined by means of induction, we arrive at our one to one mapping of A' on A. We assert, that the mapping

$$f(A) = A'$$

is topological (on A), and that A gives us the required weak realisation. f is continuous and apparently one to one on A; hence we have only to show the continuity of the mapping f^{-1}, if this time by f^{-1} is indicated the mapping of A' on A. Let

$$U_\varepsilon = U_\varepsilon\,(a_i\,|\,A)$$

be an ε-neighbourhood in A of a fixed point $a_i \in A$. We shall determine a neighbourhood

$$V' = V'(a'_i\,|\,A')$$

which is mapped under the restricted f^{-1} on a subset of U.

a'_i originates from the selection of a point in an open set being an element of a certain covering O_s.

Let

$$V' = V_{a_1a_2\ldots a_s a_{s+1}\ldots a_{s+l}} \in O_{s+l}$$

be an open set of the finite covering O_{s+l} such that

$$a'_i \in V',\; \gamma_{s+l} < {}^1\!/_2\,\varepsilon.$$

Suppose a'_k is some point of $V'\cdot A'$. a'_k originates from the selection of a point in an open set

$$V'' = V_{a_1a_2\ldots a_{s+l}\ldots a_{s+l+t}} \in O_{s+l+t}.$$

One may find one and only one finite sequence of V-sets between V' and V'' such that

$$a'_i \in V' = V_{a_1a_2\ldots a_{s+l}} \supset V_{a_1a_2\ldots a_{s+l+1}} \supset \ldots \supset V_{a_1a_2\ldots a_{s+l+t}} = V'' \ni a'_k.$$

This sequence corresponds to a fixed finite sequence of a'-points contained in the corresponding V-sets

$$a'_i, a'_{i_1}, a'_{i_2}, \ldots, a'_{i_n} = a'_k$$

such that $a'_{i_j} \in A'$ and

$$i \leqslant i_1 \leqslant i_2 \ldots \leqslant i_n = k.$$

Apparently

$$\tilde{\varrho}\,(a'_i, a'_{i_1}) < {}^1\!/_2\,\varepsilon,\ \tilde{\varrho}\,(a'_{i_1}, a'_{i_2}) < {}^1\!/_4\,\varepsilon, \ldots, \tilde{\varrho}\,(a'_{i_{n-1}}, a'_{i_n}) < \frac{\varepsilon}{2^n}.$$

From this it is clear that

$$a\,(E^i_*, E^{i_1}_*) < {}^1\!/_2\,\varepsilon,\ a\,(E^{i_1}_*, E^{i_2}_*) < {}^1\!/_4\,\varepsilon, \ldots, a\,(E^{i_{n-1}}_*, E^{i_n}_*) < \frac{\varepsilon}{2^n}.$$

From the obvious inequality

$$\min \varrho(E_*, p) \leqslant a(E_*, {}_1E_*) \text{ if } p \in {}_1E_*$$

and (4), (5), ..., we obtain

$$\varrho\,(a_i, a_{i_1}) < {}^1\!/_2\,\varepsilon,\ \varrho\,(a_{i_1}, a_{i_2}) < {}^1\!/_4\,\varepsilon, \ldots, \varrho\,(a_{i_{n-1}}, a_k) < \frac{\varepsilon}{2^n}$$

such that

$$\varrho\,(a_i, a_k) \leqslant \varrho\,(a_i, a_{i_1}) + \varrho\,(a_{i_1}, a_{i_2}) + \ldots + \varrho\,(a_{i_{n-1}}, a_k) <$$

$$< \sum_{p=1}^{n} \frac{\varepsilon}{2^p} < \sum_{p=1}^{\infty} \frac{\varepsilon}{2^p} = \varepsilon.$$

$\varrho(a_i, a_k) < \varepsilon$ means however that $a_k \in U_\varepsilon(a_i | A)$. V' is therefore mapped under (our new restricted) f^{-1} in U_ε, which we had to prove.

3.1. In our proof we established a weak *countable* realisation-set A. How far is it possible to extend the topological mapping $f(A) = A'$ under f to a topological mapping

$$f(S) = S', \quad A \subset S \subset M, \quad A' \subset S' \subset M' \text{ ?}$$

In general $S' \neq M'$, for $S' = M'$ would produce a realisation S of M' in M under f. This problem however is immediately solved by means of known theorems.

Indeed, according to a well-known theorem of LAVRENTIEFF (comp. [6], p. 214) any homeomorphism $f(A) = A'$ may be extended to a homeomorphism $g(S) = S'$

$$A \subset S \subset M = \overline{A},\ A' \subset S' \subset M' = \overline{A'}$$

where S and S' are G_δ-sets in M and M'.

Any continuous extension however of $f(A) = A'$ on a subset of M must coincide with f defined on M.

Therefore $g \equiv f$ on S. At last we observe, that the G_δ-sets S and S' are topologically-complete sets (comp. [6], p. 215).

Thus we obtain

Theorem III'. Weak realisation theorem for continuous mappings of compacta.

Any continuous mapping $f(M) = M'$ of a compactum M is topological on a suitably selected subset S of M, such that S (and therefore $f(S) = S'$) are topologically-complete sets and S' is dense in M'.

4. In 3, we have given a first simple example of a continuous mapping $f(M) = M'$ of a compactum for which no realisation (of M' in M) is possible. At first sight however one might expect that this situation is altered, if we consider a (continuous and) interior mapping, the decomposition of which is continuous. The well-known existence of dimension-raising interior mappings however makes it clear that in this case too realisations are generally impossible.

But the interior mapping $w = z^2$ of the circle $|z| = 1$ on $|w| = 1$ in the complex domain has no realisation either (although M and M', being circles, are homeomorphic), as turns out by a slight examination. In this case however the reason for the impossibility of a realisation might originate from the fact that the inverse set of an image point is not connected (consists in fact of exactly two points). Thus we arrive at continuous interior monotone mappings. Here again realisations are not possible in general, as may be shown by examples of different kind.

We give a simple but rather typical example.

Well-known is the example of BROUWER (comp. [2], [3], or [5], p. 118—120) of three simply-connected disjoint regions R_1, R_2 and R_3 (any of them therefore homeomorphic with a circle region), the boundaries of which are identical, while the sum of regions and boundary B fills up a square S.

R_1, R_2 and R_3 are mapped topologically on three circle-regions R_1', R_2' and R_3'. The collection of concentric circles filling up R_1', resp. R_2', resp. R_3' are mapped continuously on the plane sets $y = 0$, $0 \leqslant x < \frac{1}{2}$, resp. $y = 0$, $\frac{1}{2} < x \leqslant 1$ resp. $x = \frac{1}{2}$. $0 < y \leqslant \frac{1}{2}$, each circle corresponding exactly to one point. The productmapping of R_1, resp. R_2, resp. R_3 in this triod together with the mapping of B on $x = \frac{1}{2}$, $y = 0$, gives us the required mapping of S on M. This produces *an example of a continuous interior monotone mapping of a square on a triod at which no realisation is possible.*

To prove this last statement we only have to recall to mind the fact that any point of B is not accessible from R_1, R_2, and R_3. The possibility of realisation therefore breaks down at $x = \frac{1}{2}$, $y = 0$. It is however worth nothing that the inverse set B of $x = \frac{1}{2}$, $y = 0$, is not locally connected.

In (13) KNASTER gives an example of a plane irreducible continuum C which yields an interior monotone mapping of C on an interval. A realisation is apparently impossible, C being irreducible. In this example however the inverse sets of the image points are rather pathological continua. In our previous example there is one pathological inverse set B. For this reason we might not consider monotone but

Peano-monotone, interior mappings. We call a mapping Peano-monotone, if the total inverse set of any image point is a locally connected continuum. But even for Peano-monotone interior mappings realisations are generally impossible, as D. VAN DANTZIG has pointed out; this may be proved by the mapping of the space of tangential line elements on a 2-sphere on this sphere by identifying the line elements of a point with this point. The impossibility of a realisation follows from the theorem of POINCARE– BROUWER on the impossibility of a continuous field of tangential line elements on a 2-sphere.

On the other hand it seems probable to me that Peano-monotone interior mappings of *plane* sets — at least of Peano-continua — always have topological realisations. In the general case however there may arise great difficulties. Certain retraction-properties of the original sets are required. These problems are all combinatorial.

REFERENCES

1. WHYBURN, G. T., Analytic topology, Am. Math. Soc. Coll. Publ., New-York (1942).
2. BROUWER, L. E. J., Over de structuur der perfecte puntverzamelingen. Kon. Akad. v. Wetensch., Amsterdam 18, 833 — 842 (1910), 19, 1416 — 1426 (1911).
3. ————, Zur Analysis Situs, Math. Ann. 68, 422 — 434 (1910).
4. KERÉKJARTÓ, B. VAN, Vorlesungen über Topologie, Berlin (1923).
5. ALEXANDROFF, P.-HOPF, H., Topologie I, Berlin (1935).
6. KURATOWSKI, C., Topologie I, Warszwawa (1933).
7. ————, Sur les décompositions semi-continues d'espaces métriques compacts. Fund. Math. 11, 169 — 185 (1928).
8. HILL, L. S., Properties of certain aggregate functions. Bull. Am. Math. Soc. 32 (1926), Am. Journ. of Math. 49, 419 — 432 (1927).
9. MOORE, R. L., Foundations on point set theory, New York (1932).
10. STOILOW, S., Des sous-ensembles sur lesquels une transformation continue d'un espace est transformation intérieure ou topologique. Disquisit. Math. et Phys., Bucuresti 1, 23 — 28 (1940).
11. HAHN, H., Reelle Funktionen, Leipzig (1932).
12. HUREWICZ, W., Ein Satz über stetige Abbildungen. Fund. Math. 23, 54 — 62 (1934).
13. KNASTER, B., Un continu irréducible à décomposition en tranches. Fund. Math. 25, 568 — 577 (1935).

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