MATHEMATICS

EINSTEIN SPACES AND CONNECTIONS. I

BY

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In this paper we use, following CH. EHRESMANN [10, 11, 12], the terminology of the theory of fibre spaces, while dealing with spaces with a connection as defined by E. CARTAN [5, 6, 7]. This is important in view of the study of spaces in the large, but here it gives us also in the small a satisfactory view on several spaces with a connection uniquely definable with a given Riemannian, in particular Einstein-, space. Some important notions are: the fixed oblique cross section; torsion of a figure in a fibre with respect to a connection; covariant constant figures. We present two main theorems (3 and 11) on the normal conformal and the normal projective connection of an Einstein space, and several corollaries.

S. SASAKI [15, 16] kindly drew my attention to his interesting papers on Einstein spaces. I think that my presentation of the relation between the connections of an Einstein space has advantages over his method, in particular with respect to his "Poincaré's representation in the underlying manifold" (which I did not understand).

1. *The conformal group G*

Let S^n be a quadratic hyper surface of the signature of a hypersphere in a real projective $n+1$-dimensional space P^{n+1}. The projective transformations of P^{n+1} that leave S^n invariant correspond (isomorphy) with the same transformations restricted to S^n. These are if $n > 2$ the (only) twice differentiable conformal transformations of S^n.

For suitable wellknown coordinates ($-\infty < y^a < +\infty$; $a = 1, \ldots, n$), called conformally preferred (G-preferred) coordinates, covering S^n with the exception of one point, a set of operators Y_σ called infinitesimal conformal transformations are:

$$(1) \qquad Y_a = \frac{\partial}{\partial y^a}\,; \quad Y_{a\beta} = Y^a{}_\beta = -Y^\beta{}_a = y^a \frac{\delta}{\partial y^\beta} - y^\beta \frac{\partial}{\partial y^a}\,;$$

$$(2) \qquad Y_0 = y^a \frac{\partial}{\partial y^a}\,; \qquad (3) \qquad Y_{*a} = \textstyle\sum_{\beta=1}^n (y^\beta)^2 \frac{\partial}{\partial y^a} - 2 y^a y^\beta \frac{\partial}{\partial y^\beta}\,.$$

("translation; rotation; multiplication; inversion-translation-inversion"; $a, \beta = 1, \ldots, n$).

If the mapping f: part of $S^n \to n$-dimensional number space, is one

G-preferred coordinate system, then the other G-preferred coordinate systems are $f \cdot g$; $g \in G$ i.e. the group of conformal transformations of S^n.

The structure equations of the basis (1, 2, 3) of the infinitesimal conformal transformations are:

(4) $[Y_\lambda, Y_\mu] = Y_\lambda Y_\mu - Y_\mu Y_\lambda = c^\tau_{\lambda\mu} Y_\tau$ (λ, μ, τ have the same range as σ).

The constants $c^\tau_{\lambda\mu}$ are zero except those in the following equations and trivially related ones:

$$(4) \quad \begin{cases} [Y_\beta, Y_{\alpha\beta}] = [Y_\alpha, Y_0] = Y_\alpha\,; \quad [Y_{*\beta}, Y_{\beta\alpha}] = [Y_0, Y_{*\alpha}] = Y_{*\alpha}\,; \\ [Y_{\alpha\gamma}, Y_{\gamma\beta}] = Y_{\alpha\beta}\,; \quad [Y_\alpha, Y_{*\beta}] = 2\,Y_{\alpha\beta}\,; \quad \alpha \neq \beta \neq \gamma \neq \alpha\,. \end{cases}$$

A subgroup of the conformal group G operating in S^n, or of the corresponding group operating in P^{n+1}, consists for example of those group elements that leave a fixed point ψ in P^{n+1} invariant. Three possibilities may occur:

If ψ is a point of S^n, then we choose G-preferred coordinates such that ψ is the excluded or "infinite" point. The infinitesimal transformations of the subgroup has the base (1) (2). This "similarity group" has the Euclidean group G_0 with infinitesimal generators (1) as a subgroup, a subgroup, under which there exists an invariant Euclidean metric in S^n minus ψ. Those G-preferred coordinate systems for which this metric has the expression $ds^2 = \sum_{\alpha=1}^n (dy^\alpha)^2$ are called Euclidean-preferred or G_0-preferred. If f is one of them, then the others are $f \cdot g$, $g \in G_0$. Note that the metric is not (completely) determined by ψ.

If ψ in P^{n+1} is in the interior bounded by S^n (= no tangent line of S^n passes through ψ) then the polar hyper plane of ψ with respect to S^n does not intersect S^n. P^{n+1} with the exception of this hyper plane can then be covered, with preservation of topology and of straight lines, by a Euclidean space E^{n+1} such that S^n is covered by a Euclidean hypersphere of radius $\sqrt{K}$ ($K > 0$). Note that K is not determined by ψ. The subgroup (G_K) is represented by the group of motions of the space of constant curvature K introduced in S^n. For suitable G-preferred coordinates the infinitesimal transformations of G_K are those that leave invariant (the family of polarities with respect to)

$$(5) \qquad K \cdot \textstyle\sum_{\alpha=1}^n (y^\alpha)^2 + 4 = 0.$$

These coordinate systems, and also those of the next paragraph, are called non-Euclidean preferred, or more specific G_K-preferred. If f is one of these, then the others are $f \cdot g$, $g \in G_K$.

A base of the infinitesimal transformations of G_K is

$$(6) \qquad Y_{\alpha\beta}(K) = Y_{\alpha\beta}\,; \quad Y_\alpha(K) = Y_\alpha - K' \cdot Y_{*\alpha} \; (K' = K/4).$$

The invariant metric, which osculates at $y^\alpha = 0$ with the Euclidean metric $ds^2 = \sum_{\alpha=1}^n (dy^\alpha)^2$, is (E. Cartan [4] p. 164):

$$(7) \qquad ds^2 = \textstyle\sum_{\alpha=1}^n (dy^\alpha)^2 \cdot [1 + K' \cdot \sum_{\beta=1}^n (y^\beta)^2]^{-2}.$$

If ψ is exterior of S^n, then the polar hyperplane of ψ does intersect S^n in a S^{n-1} which is invariant under the subgroup. For suitable G-preferred coordinates this S^{n-1} has the equation (5) with $K < 0$. The infinitesimal transformations of the subgroup have the base (6). There is an invariant metric (7) defined in the pointset $1 + K' \cdot \sum^a (y^a)^2 > 0$. The other points of S^n are called "infinite" points.

From (4) (6) the non trivial structure equations of the Euclidean and non Euclidean groups are found to be ($K \gtreqless 0$)

$$(8) \quad [Y_a(K), Y_\beta(K)] = -K \cdot Y_{a\beta}\,; \; [Y_a(K), Y_{a\beta}] = Y_\beta(K)\,; \; [Y_{a\gamma}, Y_{\gamma\beta}] = Y_{a\beta}.$$

An infinitesimal transformation of G has an expression in any particular (G- or G_K-) preferred coordinate system. If we take another preferred coordinate system, then the same transformation will in general get another expression. We are in particular interested in the influence of those preferred coordinate transformations that yield one and the same expression for the point which has under one choosen preferred coordinate system the coordinates $y^a = 0$ (the "origin").

In the case of the G_K-preferred coordinate systems, only those coordinate transformations are then possible, that are obtained from rotations about the "origin" of the Euclidean or non-Euclidean space under consideration. Under such a coordinate transformation the operators $Y_{a\beta}$ and $Y_a(K)$ transform as a tensor and a vector of the kind indicated by the indices.

In the case of the G-preferred coordinate systems, all G-preferred coordinate transformations, also those that leave the expression of a choosen "origin" fixed, are generated by a few, the influence of which on the expression for the infinitesimal transformations is as follows:

(9)

Multiplication $\bar{y}^a = \sigma y^a$: $Y_a = \sigma \overline{Y}_a,\; Y_0 = \overline{Y}_0,\; Y_{a\beta} = \overline{Y}_{a\beta},\; \sigma Y_{*a} = \overline{Y}_{*a}.$

Rotation: The operators transform as vectors etc. of the kind indicated by their indices.

Inversion $\bar{y}^a = y^a / \sum_{\gamma=1}^n (y^\gamma)^2$: $Y_0 = -\overline{Y}_0,\; Y_{*a} = \overline{Y}_a,\; Y_a = \overline{Y}_{*a},\; Y_{a\beta} = \overline{Y}_{a\beta}.$

Translation $\bar{y}^a = y^a - \delta^a_\beta \cdot p$: $Y_0 = \overline{Y}_0 + p \cdot \overline{Y}_\beta,\; Y_{\eta\beta} = \overline{Y}_{\eta\beta} - p \cdot \overline{Y}_\eta,\; Y_{\beta\eta} = \overline{Y}_{\beta\eta} + p \cdot \overline{Y}_\eta,\; Y_{*a} = \overline{Y}_{*a} - 2p \cdot \overline{Y}_{a\beta} + p^2 \cdot \overline{Y}_a\, (a \neq \beta \neq \eta),\; Y_{*\beta} = \overline{Y}_{*\beta} - 2p \cdot \overline{Y}_0 - p^2 \cdot \overline{Y}_\beta,$ *others invariant.*

2. *Spaces with a conformal connection*

All spaces and mappings to be considered from now on, are of a sufficiently high differentiability class, say > 3. Let B and X be manifolds of dimension $2n + 1$ and n respectively. X is a neighborhood as small as we please at any moment. A mapping called *projection* is given, $p : B \to X$, such that if $x \in X$ then $p^{-1}(x)$ is an $n + 1$-dimensional manifold called fibre at x and denoted by Y^1_x.

Definition: A *reference system* is a mapping $b: B \to P^{n+1}$ such that $b \mid Y_x$ (restricted to Y_x) is topological onto P^{n+1}.

Let H ($= G$ or G_K, compare section 1) be a subgroup of the group of projective transformations in P^{n+1}. P^{n+1} assigned with the group H is called *the reference space.* Two reference systems b_1 and b_2 are called H-aequivalent, if for any $x \in X$:

$$b_2\,(b_1^{-1}\,Y_x) = h_x \in H.$$

(h_x has by assumption three derivatives with respect to x!).

We now moreover assume that B has a complete (= not contained in a larger class) H-aequivalence class of reference systems. Then B is called an *H—P^{n+1}-fibre bundle* with base space X. The reference systems of the class are called *preferred.*

Remark: If X is not restricted to be a small neighborhood, then a fibre bundle is defined by B, X, p and aequivalence classes of reference systems for neighborhoods in X.

From the definition it follows that each fibre has the structure of the reference space. The complete class of G_K-preferred reference systems of a G_K—P^{n+1}-fibre bundle is contained in one unique complete class of G-preferred reference systems. The last determines a G—P^{n+1}-fibre bundle, the *G-abstractum* of the first.

Restricting to $H = G$ or $H = G_K$ we observe that in each fibre Y_x the invariant image (under the reference system) of the quadric $S^n \subset P^{n+1}$ occurs. These images together are the pointset of an H—S^n-fibre bundle $B(H, S^n, X)$, with reference space S^n assigned with H. Vice versa $B(H, S^n, X)$ also determines $B(H, P^{n+1}, X)$.

A coordinate system for X (differentiability class > 3), a reference system of the H-aequivalence class of $B(H, S^n, X)$, and an H-preferred coordinate system for S^n (section 1), yield in a natural way a coordinate system for $B(H, S^n, X)$. A coordinate system obtained in this way will be called H-preferred.

Definition: A space X is said to possess an *a conformal connection* (or displacement) if: *A*): X is base space of a fibre bundle $B(G, S^n, X)$ $\cdot$ B): To every curve segment in X a conformal mapping is assigned of the fibre at the initial point onto the fibre at the end point of the segment. This is a continuous function of curve segments with the same initial and end point in X. C): The pseudo group of curve segments in X (addition of two curves is possible if the initial point of the second is the end point of the first curve) is homomorph onto the pseudo group of displacements. This homomorphism is obtained from the function mentioned in B) $\cdot$ D): the expression of a displacement along a differentiable curve segment in X, with respect to a preferred coordinate system for the fibre bundle, is obtained by integration from equations of the kind:

$$dy^a + \omega_i^\sigma\, dx^i\, Y_\sigma\, y^a = 0. \tag{10}$$

The range of the indices is as before. $\omega_i^\sigma = \omega_i^\sigma(x^1, \ldots, x^n)$ is a vector of the space X in the index i.

Let $x^i = x^i(u, v)$ be a differentiable surface with differentiable parameters u, v in X. The displacement along a small closed curve in X consisting of segments of parameter curves: $(u, v) = (0, 0)$, $(0, v)$, $(0, b)$, (u, b), (a, b), (a, v), $(a, 0)$, $(u, 0)$, $(0, 0)$ is approximated by

$$(11) \quad \begin{cases} \triangle y^a = (\delta d - d\delta) y^a = -2\Omega_{ki}^\sigma\, dx^k\, \delta x^i\, Y_\sigma\, y^a \\ \qquad (2\Omega_{ki}^\sigma = \partial_i \omega_k^\sigma - \partial_k \omega_i^\sigma + \omega_k^\lambda \omega_i^\mu c_{\lambda\mu}^\sigma\,;\ dx^i = \dfrac{\partial x^i}{\partial u} \cdot a\,;\ \delta x^i = \dfrac{\partial x^i}{\partial v} \cdot b). \end{cases}$$

If $\Omega_{ki}^\sigma \equiv 0$ then the space is called (locally) flat. Then the displacement along any (contractible) closed curve in X, is the identity mapping of the fibre at the initial = end point. Ω_{ki}^σ is a tensor in X with respect to the indices k, i.

A change of reference system for the fibre bundle carries with it self a change of coordinates for each fibre. In one fibre, but not in all at the same time, this change of coordinates by virtue of the reference transformation, can be annihilated by a preferred coordinate transformation of the reference space S^n. The influence of some preferred coordinate transformations of S^n on $\Omega^\sigma\ (= \Omega_{ki}^\sigma)$, is as follows:

$$(12) \quad \begin{cases} \textit{Multiplication } \bar{y}^a = \sigma \cdot y^a\colon\ \bar{\Omega}^a = \sigma\,\Omega^a,\ \sigma\,\bar{\Omega}_{*a} = \Omega_{*a},\ \bar{\Omega}^0 = \Omega^0,\ \bar{\Omega}^{a\beta} = \Omega^{a\beta}. \\ \textit{Rotation: } \Omega^a, \Omega^{*a}, \Omega^{a\beta} \textit{ transform as vectors of the kind indicated by} \\ \textit{the indices} \cdot \bar{\Omega}^0 = \Omega^0. \\ \textit{Inversion: } \bar{\Omega}^0 = -\Omega^0,\ \bar{\Omega}^a = \Omega^{*a},\ \bar{\Omega}^{*a} = \Omega^a,\ \bar{\Omega}^{a\beta} = \Omega^{a\beta}. \\ \textit{Translation: see } (9).\quad \bar{\Omega}^0 = \Omega^0 - 2p \cdot \Omega^{*\beta},\ \bar{\Omega}^{*a} = \Omega^{*a}, \\ \qquad \bar{\Omega}^{a\eta} = \Omega^{a\eta} - p \cdot \delta_\beta^\eta\, \Omega^{*\eta} + p\, \delta_\beta^a\, \Omega^{*a}, \\ \qquad \bar{\Omega}^a = \Omega^a - 2p\,\Omega^{a\beta} + p^2 (1 - 2\delta_\beta^a)\, \Omega^{*a} + p\, \delta_\beta^a\, \Omega^0. \\ \textit{Inversion-translation-inversion: } \\ \qquad \bar{\Omega}^0 = \Omega^0 + 2p\,\Omega^\beta,\ \bar{\Omega}^a = \Omega^a, \\ \qquad \bar{\Omega}^{a\eta} = \Omega^{a\eta} - p\, \delta_\beta^\eta\, \Omega^\eta + p\, \delta_\beta^a\, \Omega^a, \\ \qquad \bar{\Omega}^{*a} = \Omega^{*a} - 2p\,\Omega^{a\beta} + p^2 (1 - 2\delta_\beta^a)\, \Omega^a - p\, \delta_\beta^a\, \Omega^0. \end{cases}$$

The space with a conformal (G-) connection which we defined sofar, needs much additional structure, before a space with a conformal connection in the customary sense (CARTAN E.) is obtained.

First Assumption. With each fibre is associated one fixed point in that fibre, such that these points form a n-dimensional sub manifold of the fibre bundle. We call this pointset *the fixed cross section*. The point in the fibre represents the corresponding point in the base space X. We choose the preferred reference system of the fibre bundle and the preferred coordinate system of the reference space such, that the point of the fixed cross section in each fibre has coordinates $y^a = 0$ ("origin").

Definition: A point with coordinates say y^a in the fibre Y_x at the point x of X is said to be without torsion if equation (11) yields

$$\text{at } x : \triangle y^a = 0. \tag{13}$$

Under a displacement along a small circuit in the base space X, such a point returns in first approximation to its original place. It is easily seen that the origin ($y^a = 0$) of the fibre Y_x is without torsion, if and only if Ω^a (at x) $= 0$ ($a = 1, \ldots, n$).

Second Assumption. All points of *the fixed cross section* are *without torsion* with respect to the given connection:

$$\Omega^a(x) = 0 \qquad (a = 1, \ldots, n \; ; \; x \in X). \tag{14}$$

(This is usually expressed by the words: the connection is without torsion. Note that also the vanishing of the torsion of any figure, instead of a point, in a fibre can be defined in a similar way.)

The influence of reference transformations which leave the expression of the "origin" invariant, on the numbers Ω^a_{ki} of the space with a conformal connection without torsion is as follows (Compare (12)):

$$(15)\quad \begin{cases} \textit{Similarity}: \ \overline{\Omega}^a = \Omega^a = 0,\ \overline{\Omega}^{*a} = \sigma^{-1} \cdot \Omega^{*a}, \ \textit{others invariant.} \\ \textit{Rotation}: \textit{as before } (12). \\ \textit{Inversion-translation-inversion (Compare (9))}: \\ \qquad \overline{\Omega}^{*a} = \Omega^{*a} - 2p\, \Omega^{a\beta} - p\, \delta^a_\beta\, \Omega^0, \ \textit{others invariant.} \end{cases}$$

Consider the points of a curve segment in X, and also the points of the fibres which by virtue of the fixed cross section represent these points. Displacement along any segment with end point x_0 of the curve, yields a representation of the initial point of that segment in the fibre at x_0. These points in the fibre at x_0 which correspond with points of the curve form by definition *the development of the curve in that fibre* with respect to the connection and the fixed cross section. This development is a curve in the fibre at x_0 which passes through the "origin".

If the curve $x^i = x^i(t)$ in X is differentiable with parameter t, then the tangent to the development in the fibre at $x^i = x^i(0)$, at the origin, has the direction $\omega^a_i.\ dx^i/dt$ if this is different from 0. If this direction exists ($\neq 0$) for every choice of $dx^i/dt \neq 0$ at the point x_0 to be considered, then the fixed cross section is called oblique with respect to the connection at the point $x_0 \in X$. A necessary and sufficient condition is: determinant $\omega^a_i \neq 0$.

Third Assumption. The fixed *cross section is oblique* with respect to the connection at all points of X.

Let Y_x be a fibre of the fibre bundle $B(G, S^n, X)$ of a space with a G-connection, and let χ be a point of Y_x. Consider the n-dimensional space

$T(\chi)$ of tangent vectors at χ with respect to the fibre. The angle between two of these vectors is defined by virtue of the reference system, which is a mapping onto S^n. From $T(\chi)$ a Euclidean n-dimensional space can be obtained, by assigning a number (length) to one of the vectors. The other vectors then also have a natural length. Such an "imbedding" of a Euclidean space in $T(\chi)$ is called a *gauge* at χ.

Now let χ be without torsion. Let the fibre Y_x be displaced along a small closed parallellogram as above. Considering first order terms only, let all gauges at χ be invariant under this displacement. If the same is true for any small closed parallellogram with vertix x in X, then the gauges at the point χ of the fibre Y_x are said to be without torsion.

Fourth Assumption. The gauges at every point of the fixed cross section are without torsion. This condition is expressed by the invariant equations

$$\Omega^0_{ki} = 0. \tag{16}$$

The first three assumptions are also made in the cases of spaces with a G_K-connection, and they have the same expression (Compare section 1). The fourth assumtion need not be stated because it is true anyhow in this case.

$\omega^a_i \dfrac{dx^i}{dt}$ $(x = x_0)$ is a vector of the fibre at x_0. It is a vector at the point of the fixed cross section of this fibre. If we define $\Omega^a_{\beta ij} = \Omega^{a\beta}{}_i$ then $\Omega^a_{\beta ij}$ is a tensor in the indices a, β at the same point of the fibre (Compare (14)). We now define numbers ω^i_a by

$$\omega^i_a \omega^\beta_i = A^\beta_a \quad \text{hence} \quad \omega^i_a \omega^a_j = A^i_j \quad (a, \beta, i, j = 1, \dots n). \tag{17}$$

$A^a_\beta = 0$ if $a \neq \beta = 1$ if $a = \beta$; A^i_j analogous.

ω^i_a is a covariant vector with respect to the fibre in the index a and a contravariant vector with respect to the base space in the index i.

From the influence of preferred reference transformations (with a fixed expression $y^a = 0$, for the fixed point in each fibre) on $\Omega^a_{\beta ij}$, ω^a_i, ω^i_a, it now follows, that the following tensors with respect to the base space, are invariant under these reference transformations of the fibres, and therefore these entities are just tensors with respect to the base space:

The conformal curvature tensor

$$\Omega^i_{jkm} = \Omega^a_{\beta km}\, \omega^i_a\, \omega^\beta_j\,. \tag{18}$$

The conformal Ricci tensor

$$\Omega^i_{jim}\,. \tag{19}$$

If we consider instead of the conformal group G, the subgroup G_K, then we get by definition a space with a Euclidean or Non-Euclidean connection. Instead of (11) we then have:

$$\triangle y^a = -\,2\,\Omega^\sigma_{ki}(K)\, dx^k\, \delta x^i\, Y_\sigma(K)\, y^a \tag{20}$$

where σ has now only the range $\sigma = a,\, a\beta$ $(a, \beta = 1, \dots n;\, a \neq \beta)$.

A G_K-connection in a G_K—S^n-fibre bundle is carried over in a natural way to the G-abstractum of this fibre bundle. The space with a G-connection obtained in this way is called the *G-abstractum (conformal abstractum) of the given space with a G_K-connection.* The conformal curvature Ω^{σ}_{ki} of this abstractum of the space with curvature $\Omega^{\sigma}_{ki}(K)$ obeys:

$$\left\{ \begin{array}{l} \Omega^{a}_{ki} = \Omega^{a}_{ki}(K) = 0, \ \Omega^{*a}_{ki} = -K\,\Omega^{a}_{ki}(K) = 0 \\ \Omega^{0}_{ki} = 0\,; \ \Omega^{a}_{\beta ki} = \Omega^{a}_{\beta ki}(K). \end{array} \right. \tag{21}$$

Hence:

$$\Omega^{i}_{jkm} = \Omega^{i}_{jkm}(K)\,; \ \Omega^{i}_{jik} = \Omega^{i}_{jik}(K) \tag{22}$$