MATHEMATICS

EINSTEIN SPACES AND CONNECTIONS. II

BY

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3. *Einstein spaces*

Let a Riemannian metric be given in the space X: $ds^2 = g_{ij} dx^i dx^j$. Then a unique space with a Euclidean connection and with base space X is determined by, apart from the assumptions of section 2, the properties: The development of any differentiable curve $x_i = x_i(t)$ in the fibre at one of its points, has the same length as the curve itself; the development of a broken curve the two parts of which meet under an angle a, consists of two parts that meet under the same angle a (LEVI–CIVITA [14]; SCHOUTEN [17]: parallellism). We recall the equation

$$\sum_{a=1}^{n} \omega_i^a \omega_j^a = g_{ij}. \tag{23}$$

Let us now consider at each point of X a space of constant curvature K, osculating at the fixed point of the Euclidean fibre with this Euclidean space. The G_K-connection defined by the same numbers ω_i^a, $\omega_i^{a\beta}$ as those to be found in the unique way mentioned above (with respect to given preferred coordinates) has the same properties that we mentioned for the Euclidean connection. It is therefore the unique G_K-connection (non-Euclidean connection) with these properties. The curvature of the two connections are related as follows (Compare (8) and (11)).

$$\Omega^a_{\beta km}(K) = \Omega^a_{\beta km}(0) - K(\omega_k^a \omega_m^\beta - \omega_m^a \omega_k^\beta). \tag{24}$$

Also, after multiplication with $\omega_a^i \omega_j^\beta$ and summation over $a, \beta = 1, \ldots, n$,

$$\Omega^i_{jkm}(K) = \Omega^i_{jkm}(0) - K(\delta^i_k g_{jm} - \delta^i_m g_{jk}). \tag{25}$$

Hence:

$$\Omega^i_{jik}(K) = \Omega^i_{jik}(0) - K(n-1) g_{jk}. \tag{26}$$

From (25) follows that the tensor of the non-Euclidean connection of a Riemannian space obeys identities analogous to the well known identities of *the* Riemann tensor:

$$\Omega_{ijkm}(K) \overset{\text{def}}{=} g_{ip} \Omega^p_{jkm}(K); \; \Omega_{ijkm}(K) = -\Omega_{jikm}(K) = \Omega_{kmij}(K). \tag{27}$$

A Riemannian space is called an Einstein space of scalar curvature K if the right hand side of (26) vanishes. Hence it is *characterised by*

Theorem 1. The G_K-connection of an Einstein space of scalar curvature K has a vanishing G_K-Ricci tensor.

A space of constant curvature K has the characteristic property that the right hand side of (25) vanishes. It is *characterised by*

Theorem 2. The G_K-connection of a space of constant curvature K is (locally) flat.

The conformal structure of a Riemannian space determines a unique conformal connection (CARTAN [5]), the so called *normal conformal connection*, determined by, apart from the assumptions of section 2, the property: The conformal Ricci-tensor vanishes

$$\Omega^{i}_{jim} = 0. \tag{28}$$

It seems to be difficult to understand geometrically the contents of the equations (28). The normal conformal connection of an Einstein space however can be characterised in a satisfactory geometrical way as follows: Theorem 1, (21), (22) and (27) yield

Main theorem 3. *The normal conformal connection of an Einstein space of scalar curvature K is the conformal abstractum of the G_K-connection of this space.*

Theorem 4 (SCHOUTEN, STRUIK [18]). *A conformally flat Einstein space is a space of constant curvature.*

Proof: Let the scalar curvature of the Einstein space be K. Then the G_K-connection is flat. (Th. 3).

Theorem 5 (SCHOUTEN, STRUIK [18]). *A three dimensional Einstein space is of constant curvature.*

Proof: The conformal curvature tensor of the normal conformal connection of a three dimensional Riemannian space vanishes. Let the scalar curvature of the given Einstein space be K. Then the G_K-connection of this space is (locally) flat (Th. 2 and 3).

In section 1 we introduced the conformal group of transformations in $S^n \subset P^{n+1}$. All conformal, Euclidean and non-Euclidean fibres can be assumed to be imbedded in the way of section 1 in P^{n+1}'s. Also: the spaces with a conformal etc. connection determine (extend) in a unique way (to) spaces with a connection and as fibres P^{n+1}'s. We call those connections *conformal P^{n+1}-connections*. If the connection is obtained from a normal conformal connection, then we call it also *normal*.

In the case of a Euclidean or non-Euclidean connection, the in this way constructed space with a P^{n+1}-connection, has in each fibre one point of particular interest: the point ψ of section 1. These points map onto each other under displacements along arbitrary curves in the base space.

They form a cross section, which for these reasons is called *a covariant constant point* of the conformal P^{n+1}-connection. From th. 1 and 3, we now have:

Theorem 6 (SASAKI [15]). *The normal conformal P^{n+1}-connection of an Einstein space has a covariant constant point, which lies outside, on, inside the covariant constant S^n in case the scalar curvature K is $<, =, >$ 0 respectively.*

Example: DEBEVER [9] proved, without knowing this theorem of SASAKI, its application to the group space of a semi simple group which has an Einstein metric according to CARTAN and SCHOUTEN [8].

Conversily, if a normal (!) conformal P^{n+1}-connection has a covariant constant point, then this point can be used to introduce a covariant constant metric in the covariant constant S^n or part of it. The metric and the connection, determine a G_K-connection of which the G_K-Ricci tensor vanishes. This connection introduces a metric in the base space, except in those points in the fibres of which the fixed cross section intersects the "infinite" point(s) of the fibre, infinite with respect to the introduced metric in the fibre S^n. Those points of the base space have to be excluded, because their distance to ordinary points cannot be defined properly. We have:

Theorem 7 (SASAKI [15]). *If the normal conformal P^{n+1}-connection of a Riemannian n-dimensional space V has a covariant constant point in the interior, on, in the exterior of the covariant constant S^n, then V is conformal to an Einstein space with (constant) positive, zero and negative scalar curvature respectively. In the last two cases it may happen that some "infinite" points of the base space have to be excluded.*

In the case of theorem 7 it is obvious that the given Riemannian space is conformal to a set of Einstein spaces each of which is obtained from one of them by a multiplication of all distances with a positive factor. This corresponds with different choices of the metric in the $S^n \subset P^{n+1}$. Compare section 1; with a point ψ in P^{n+1} correspond several metrics in S^n.

V is said to admit a conformal mapping onto k *conformally independant Einstein spaces*, if its normal conformal P^{n+1} connection has k covariant constant points not contained in a (covariant constant) projective subspace of dimension $< k-1$.

A set of covariant constant figures, e.g. points, S^n, a non-Euclidean metric in S^n, determines the subgroup of those projective transformations in one fibre (P^{n+1}) that leave the representative figures invariant. All figures in this fibre, invariant under this subgroup, are then also representative for covariant constant figures. If the subgroup is the identity then the connection is flat.

Suppose for example that a conformal P^{n+1} connection has two covariant constant points in the interior of S^n. Then all points of the line passing through these two points are also covariant constant (consider representative figures in one fibre).

A covariant constant figure is in each fibre represented by a figure which is a fortiori without torsion. Sometimes it is also known that in each fibre another figure is without torsion e.g. the point of the fixed cross section, or the gauges there. Other figures without torsion may then be found, eventually all points of each fibre in which case the connection is flat.

These are the ways to prove theorems like:

Theorem 8. *If a Riemannian space is conformal to two conformally independant Einstein spaces, then the space is certainly conformal to an Einstein space of negative scalar curvature* (Eventually some "infinite" points excluded); *If one of the given Einstein spaces has positive scalar curvature, then the given space is also conformal to an Einstein space with vanishing scalar curvature.* (Compare SASAKI [15], BRINKMAN [1, 2, 3]. The proof is left to the reader.

Theorem 9 (SASAKI [15]). *If an Einstein space V of dimension n and scalar curvature K admits a conformal mapping onto $n-2$ conformally independant Einstein spaces* (itself included), *then it is a space of constant curvature K.*

Proof: Consider the G_K-connection of the given space. It is easy to check that the Euclidean ($K=0$) or non-Euclidean ($K \neq 0$) fibre contains at least $n-3$ independant covariant constant points ("infinite" points are not counted). Suppose that in a fibre to be considered the point of the fixed cross section is independant of the points representing the covariant constant points *. Then at the point of the fixed cross section we find $n-3$ independant directions without torsion. If we choose favourable preferred coordinates for that fibre, and coordinates in the base space such that ω_i^a (at that point of the base space) equals $\delta_i^{a'}$, then

(29) at $x = x_0$, $\Omega^i_{jkm} = \Omega^i_{jkm}(K) = 0$ if at least one of the indices > 3.

(27), (28) and (29) give $\Omega^i_{jkm}(K) = 0$ at $x = x_0$, $i, j, k, m = 1, \dots n$.
The same holds for all points x for which * holds, and these points are every where dense in V, so that continuity of $\Omega^i_{jkm}(K)$ leads to the theorem.

We conclude this section with a theorem in the large. A Riemannian space is called (metrically-)*complete*, if any segment of a geodesic is contained in a segment with the same begin point and one unit longer. The G_K-connection of the Riemannian space has the characteristic property, that any curve segment in one fibre, passing through the point of the fixed cross section, which does not contain "infinite" points of the fibre, *is* the development of some curve in the base space. Suppose the given space is an Einstein space of non-positive scalar curvature. Consider the normal conformal P^{n+1} connection. From the completeness of the given space and the theorem 3 follows that the "infinite" covariant constant point(s) in the covariant constant S^n are determined by the normal conformal connection and the base space as point set. Hence the point ψ of section 1 is determined by the conformal structure. The metric in each fibre is determined by ψ, but for the choice of a unit of distance. The same holds true for the metric in the base space. Therefore:

Theorem 10. *A conformal mapping of a metrically complete Einstein space of non-positive scalar curvature onto another metrically complete Einstein space is the product of a (locally-)congruent mapping and a multiplication of the metric* (ds^2) *with a constant positive factor.*

Example: A conformal mapping of the Euclidean space onto itself, or onto a locally Euclidean n-dimensional torus.

4. *Einstein spaces and projective connections*

If: the reference space for the fibres of a fibre bundle with an n-dimensional base space X, is an n-dimensional projective space; the group analogous to H in section 2 is the group of projective transformations; and a displacement in the fibre bundle is defined as in section 2; then a space with a projective connection is defined. The existence of a fixed oblique cross section without torsion is assumed.

Let us consider the reference space P^n. The Euclidean space is obtained from P^n by a choice of: a hyperplane P^{n-1} of excluded "infinite" points in P^n; a polarity with respect to an imaginary quadric in P^{n-1}; a unit of distance in P^n minus P^{n-1}.

The non-Euclidean space of constant negative curvature is obtained by a choice of: a real quadratic hypersurface S^{n-1} of the signature of a sphere in P^n; a unit of distance for the interior of this sphere. (The other points of P^n are called "infinite" points.)

The non-Euclidean space of constant positive curvature is obtained by a choice of: an imaginary quadratic hypersurface in P^n; a unit of distance.

Vice versa these spaces can always be considered to be imbedded in the described way in a uniquely determined P^n. Summarising, in all three cases there is an (eventually degenerated) hypersurface of the second class, and a unit of distance.

Because the Euclidean and non-Euclidean groups (G_K) are subgroups of the group P of projective transformations operating in the same space or a uniquely determined space obtained by addition of some "infinite" points, a definition of the *projective abstractum of a Euclidean or non-Euclidean connection* can be given, analogous to the conformal abstractum of section 3.

The system of geodesics of a Riemannian space determines one unique "normal" projective connection. This connection obeys apart from the conditions allready mentioned: a condition which has analogy with the fourth assumption (16) of section 2; and a condition of vanishing of a projective Ricci-tensor analogous to (28). (CARTAN [7]). It is difficult to understand geometrically this normal projective connection. For Einstein spaces of scalar curvature K however a satisfactory geometrical characterisation can be given:

Main theorem 11. *The normal projective connection of an Einstein space of scalar curvature K is the projective abstractum of the G_K-connection.*

The proof is analogous to the proof of main theorem 3.

The equality of the so called (normal) projective curvature tensor P^i_{jkm}, the (normal) conformal curvature tensor C^i_{jkm} and the G_K-curvature tensor (25) of an Einstein space (K), can easily be checked from the formulas for P^i_{jkm} and C^i_{jkm}. (EISENHART [13], SCHOUTEN–STRUIK [19])

As a corollary of theorem 11 we have for example:

Theorem 12 (SASAKI–YANO [16]. *The normal projective connection of an Einstein space has a covariant constant hypersurface of the second class.* If the scalar curvature is K, then the hypersurface has the normalised equation in homogeneous hyperplane coordinates for one fibre P^n

$$\sum_{i=1}^{n} \xi_i^2 + K\, \xi_{n+1}^2 = 0. \tag{30}$$

Vice versa: if the normal projective connection of a Riemannian space V does have a covariant constant figure of the kind (30), then the connection has a covariant constant Euclidean or non-Euclidean metric in the fibres, with the help of which an agreeing Einstein metric in the base space can be introduced. V then admits a mapping with preservation of geodesics (a projective mapping) onto that Einstein space (SASAKI [16]). As with the analogous theorem on conformal connections, it may happen that some points, the "infinite" points, have to be excluded from the new space. Here however worse may happen: In case $K < 0$, the theorem does not hold, if all points of the fixed cross section are exterior of the covariant constant (infinite) S^n.

The proof of the next theorem is representative for the proofs of a class of theorems. If the Einstein spaces which are given have non-vanishing scalar curvature, then a fairly simple proof can be given. The "exceptional" cases when some of the Einstein spaces have vanishing scalar curvature make the proofs lengthy.

Theorem 13. *If a fourdimensional Einstein space admits an essentially projective mapping onto another four dimensional Einstein space, then it is a space of constant curvature.*

A mapping is here called essentially projective, if it is not the product of a congruent mapping and a multiplication of all distances with a constant factor; or, what amounts to the same, if the two metrics of the space with the common normal projective connection realise by different covariant constant hypersurfaces of class two. (Note that we deal with local properties, that is with a small neighborhood of the base space, so that topological difficulties do not occur here).

Proof: Let the scalar curvatures of the two Einstein spaces be K and K'.

Case A: $K \neq 0$, $K' \neq 0$. The normal projective connection has two covariant constant non-degenerated hyperquadrics. Also the pencil of hyperquadrics with these two as basis, is covariant constant. Consider a fibre in which the point of the fixed cross section is not contained in any of the two hyperquadrics *. By assumption this point is without torsion. Also the hyperquadric of the pencil which passes through this point (in the fibre under consideration) is then without torsion, and the same is true for the hyperplane (of dimension 3) tangent at this point to the hyperquadric. We assume the existence of this hyperplane *. In view of main theorem 11, the G_K-connection of the first Einstein space has at the point of the base space under consideration a threedimensional hyperplane through the point of the fixed cross section *without torsion*. The direction perpendicular to this hyperplane at the same point is also without torsion. The rest of the proof is the same as in the proof of theorem 9.

Case B: $K \neq 0$, $K' = 0$. The normal projective connection has a covariant constant non degenerated hyper quadric and a covariant constant hyperplane. The hyperplane counted twice may serve as a second hyperquadric. The rest of the proof is as in case A.

Case C: $K = 0$, $K' = 0$. If the two covariant constant hyperplanes do not coincide, then they are the base of a pencil of covariant constant hyperplanes. The rest of the proof is then as before. Now suppose that the covariant constant hyperplanes coincide. The covariant constant imaginary quadrics in this hyperplane have for suitable homogeneous coordinates the equations:

$$\left\{\begin{array}{l} x^2+y^2+z^2+t^2=0, \quad a x^2+b y^2+c z^2+d t^2=0, \\ a, b, c, d>0, \quad a+b+c+d=4. \end{array}\right. \tag{31}$$

They determine invariantly (hence also covariant constant) a point in case the four numbers a, b, c, d are not equal in pairs. The covariant constant point determines in almost every fibre a line without torsion: the line through this point and the point of the fixed cross section. The rest of the proof is as before.

In case the numbers a, b, c, d are equal in pairs (they are not all mutually equal, because then the given mapping would not be essential) the normal projective connection has two covariant constant lines in the covariant constant ("infinite") hyperplane. The Euclidean (G_0) connection of say the first space ($K = 0$) has two perpendicular covariant constant two-directions. For suitable coordinates, for which $g_{ij} = \delta_{ij}$, $\partial_i g_{jk} = 0$ at

$x = x_0$, in a neighborhood of a point x_0 of the base space, the (ordinary !) Riemann tensor then obeys, apart from

$$\textstyle\sum_{j=1}^{4} \Omega_{ijjk} = \Omega_{1234} = 0 \quad \text{also} \quad \Omega_{31\,km} = \Omega_{41\,km} = \Omega_{32\,km} = \Omega_{42\,km} = 0,$$

hence this tensor vanishes. q.e.d.

The generalisation to higher dimensions (compare theorem 9) of theorem 13 holds true and is easy to prove if the Einstein spaces under consideration have non-vanishing scalar curvature. Otherwise the proofs get complicated by the large number of details.

We conclude section 4 with the statement of a theorem analogous to theorem 10

Theorem 14. *A projective mapping of a metrically complete Einstein space of negative* (*vanishing*) *scalar curvature onto another such Einstein space is the product of a* (*locally-*)*congruent mapping and a multiplication of all distances with a constant factor* (*is an affine mapping, that is: it preserves ratios of lengths of segments of any geodesic*).

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