MATHEMATICS

AN ARITHMETICAL THEOREM CONCERNING LINEAR DIFFERENTIAL EQUATIONS OF INFINITE ORDER

BY

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It is a matter of common knowledge that analysis often plays an important part in deriving purely arithmetical results. However, by combining in the same manner analytical methods and ideas from the theory of numbers, one is often led to theorems of mixed arithmetical and analytical character. The theorems derived in this paper are of this type.

In order to state our principal result, it is convenient to introduce first for a given integral function $y(z)$ the notion of an "*exceptional point*"; we shall call a complex number ζ an exceptional point for the function $y(z)$ if both values ζ and $y(\zeta)$ are algebraic numbers. If μ is a positive integer, such that ζ, $y(\zeta)$, $y'(\zeta),\dots, y^{(\mu-1)}(\zeta)$ all are algebraic, but $y^{(\mu)}(\zeta)$ transcendental, then μ will be called the "*multiplicity*" of the exceptional point ζ. If possibly ζ and all values $y(\zeta)$, $y'(\zeta),\dots$ are algebraic, then the multiplicity of μ will be infinite by definition.

Theorem I. *Let the integral function*

$$y(z)=\sum_{h=0}^{\infty} c_h \frac{z^h}{h!} \quad , \quad \limsup_{h\to\infty} \sqrt[h]{|c_h|} \leqq q,$$

where q denotes an arbitrary positive number and all coefficients $c_0, c_1, c_2,\dots$ are algebraic, satisfy a linear differential equation of infinite order

$$a_0\, y(z)+a_1\, y'(z)+a_2\, y''(z)+\dots=0,$$

with constant coefficients $a_0, a_1, a_2,\dots$ not vanishing simultaneously. Let the corresponding characteristic function

$$a_0+a_1\, t+a_2\, t^2+\dots$$

be regular in the circle $|t|\leqq q$ and let ν denote the maximum of the multiplicities of its zeros in the region $0<|t|\leqq q$.

Then the following two assertions are true:

1. *If $y(z)$ has ν or more exceptional points different from zero* (counting a point of multiplicity μ also μ-times), *then $y(z)$ necessarily is a polynomial with algebraic coefficients.*

2. *If* $a_0 \neq 0$, *then every exceptional point different from zero and with multiplicity* μ *necessarily is a zero of* $y(z)$ *with the same multiplicity* μ. (This assertion still holds if μ is infinite.)

Remark. It follows from assertion 1: If the *transcendental* function $y(z)$ fulfills the conditions of our theorem and if moreover the zeros of the characteristic function $a_0 + a_1 t + a_2 t^2 + \ldots$ are *simple* (hence $\nu = 1$ in the preceding theorem), then $y(\zeta)$ is transcendental for every algebraic value of $\zeta \neq 0$.

We give a proof of this theorem in § 1; some of the ideas we use are due to McMillan, who stated a theorem closely related to ours in an earlier paper [1]). However the result obtained by McMillan is erroneous as will be shown in § 2. In § 3 we give some interesting applications of our theorem I. In this manner we obtain three known theorems, respectively due to Itihara, to Dietrich and Rosenthal and to R. Rado. Moreover we find a new result concerning linear differential-difference equations (theorem II). The paper closes with some references.

§ 1. The principal tools we need for the proof of theorem I are:

a) The Lindemann–Weierstrass theorem: *Let* $\alpha_1\, \alpha_2, \ldots, \alpha_n$ *denote different algebraic numbers, let* $\beta_1, \beta_2, \ldots, \beta_n$ *denote arbitrary algebraic numbers. If*

$$\beta_1 e^{\alpha_1} + \beta_2 e^{\alpha_2} + \ldots + \beta_n e^{\alpha_n} = 0,$$

then necessarily $\beta_1 = \beta_2 = \ldots = \beta_n = 0$.

b) The analogous but elementary theorem: *Let* $\varrho_1, \varrho_2, \ldots, \varrho_n$ *denote different numbers, let* $P_1(z), P_2(z), \ldots, P_n(z)$ *denote arbitrary polynomials. If*

$$P_1(z)\, e^{\varrho_1 z} + P_2(z)\, e^{\varrho_2 z} + \ldots + P_n(z)\, e^{\varrho_n z} \equiv 0,$$

then necessarily $P_1(z) \equiv P_2(z) \equiv \ldots \equiv P_n(z) \equiv 0$.

c) A theorem essentially due to Schürer [2]): *Let the integral function*

$$y(z) = \sum_{h=0}^{\infty} c_h \frac{z^h}{h!} \quad , \quad \limsup_{h\to\infty} \sqrt[h]{|c_h|} \leqq q,$$

satisfy a linear differential equation of infinite order

$$a_0\, y(z) + a_1\, y'(z) + a_2\, y''(z) + \ldots = 0,$$

with constant coefficients not vanishing simultaneously. Let the characteristic function

$$A(t) = a_0 + a_1 t + a_2 t^2 + \ldots$$

be regular for $|t| \leqq q$.

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[1]) For references see the list at the end of this paper.

[2]) See also the papers of Perron and Sheffer.

If $A(t)$ has no zeros in the circle $|t| \leqq q$, then necessarily $y(z) \equiv 0$.

In all other cases there exists a polynomial $b_0 + b_1 t + \ldots + b_k t^k$ with zeros (also with respect to their multiplicities) *identical with those of $A(t)$ in the circle $|t| \leqq q$. Then $y(z)$ satisfies the linear differential equation of finite order*

$$(1) \qquad b_0\, y(z) + b_1\, y'(z) + \ldots + b_k\, y^{(k)}(z) = 0.$$

From the theorems *b*) and *c*) we deduce the following lemma:

Lemma: *Let the integral function*

$$y(z) = \sum_{h=0}^{\infty} c_h \frac{z_h}{h!} \not\equiv 0 \quad , \quad \limsup_{h\to\infty} \sqrt[h]{|c_h|} \leqq q,$$

with algebraic coefficients $c_0, c_1, c_2, \ldots$ satisfy a linear differential equation of infinite order

$$a_0\, y(z) + a_1\, y'(z) + a_2\, y''(z) + \ldots = 0,$$

with constant coefficients not vanishing simultaneously and such, that the characteristic function

$$a_0 + a_1\, t + a_2\, t^2 + \ldots$$

is regular for $|t| \leqq q$.

Then $y(z)$ can be written

$$y(z) = \sum_{i=1}^{j} P_i(z)\, e^{\varrho_i z}.$$

Here $\varrho_1, \varrho_2, \ldots, \varrho_j$ represent different algebraic numbers, zeros of the characteristic function $a_0 + a_1\, t + a_2\, t^2 + \ldots$ in the circle $|t| \leqq q$; moreover every $P_i(z)$ is a polynomial with algebraic coefficients and of degree $\nu_i - 1$ at most, ν_i denoting the multiplicity of the zero ϱ_i $(i = 1, 2, \ldots, j)$.

Proof. 1. The function $y(z)$ considered here satisfies all the hypotheses of SCHÜRER's theorem *c*). Now $y(z) \not\equiv 0$, hence the characteristic function $a_0 + a_1\, t + a_2\, t^2 + \ldots$ must have zeros in the circle $|t| \leqq q$. Let $\varrho_1, \varrho_2, \ldots, \varrho_s$ represent these zeros and let $\nu_1, \nu_2, \ldots, \nu_s$ denote their respective multiplicities. Let $b_0 + b_1\, t + \ldots + b_k t^k$ be a polynomial with zeros $\varrho_1, \varrho_2, \ldots, \varrho_s$ of multiplicities $\nu_1, \nu_2, \ldots, \nu_s$ $(b_k \neq 0)$. Then, on account of SCHÜRER's theorem, $y(z)$ satisfies

$$(2) \qquad b_0\, y(z) + b_1\, y'(z) + \ldots + b_k\, y^{(k)}(z) = 0 \qquad (b_k \neq 0).$$

Hence $y(z)$ can be written

$$(3) \qquad y(z) = \sum_{\sigma=1}^{s} P_\sigma(z)\, e^{\varrho_\sigma z},$$

where every $P_\sigma(z)$ represents a polynomial of degree $\nu_\sigma - 1$ at most $(\sigma = 1, 2, \ldots, s)$.

2. Now we shall use the condition, that all coefficients $c_0, c_1, c_2, \ldots$ of $y(z)$ are algebraic. In stead of (2) we may write

$$(4) \qquad L\,[y(z)] \equiv 0,$$

if we introduce the linear differential operator

$$L = b_0 + b_1 D + \dots + b_k D^k.$$

The $k+1$ numbers $b_0, b_1, \dots, b_k$ have a linear independent basis $\tau_1, \tau_2, \dots, \tau_r$ with respect to the field of algebraic numbers; hence

$$b_\varkappa = b_{\varkappa 1}\tau_1 + b_{\varkappa 2}\tau_2 + \dots + b_{\varkappa r}\tau_r \qquad (\varkappa = 0, 1, 2, \dots, k) \tag{5}$$

with algebraic $b_{\varkappa 1}, b_{\varkappa 2}, \dots, b_{\varkappa r}$. It follows

$$L = \tau_1 L_1 + \tau_2 L_2 + \dots + \tau_r L_r, \tag{6}$$

if we put

$$L_\varrho = b_{0\varrho} + b_{1\varrho} D + \dots + b_{k\varrho} D^k \qquad (\varrho = 1, 2, \dots, r).$$

Now all coefficients in $y(z) = c_0 + c_1 \frac{z}{1!} + c_2 \frac{z^2}{2!} + \dots$ are algebraic; also the coefficients $b_{0\varrho}, b_{1\varrho}, \dots, b_{k\varrho}$ in the above operator L_ϱ are algebraic. It follows easily

$$L_\varrho [y(z)] = c_{0\varrho} + c_{1\varrho}\frac{z}{1!} + c_{2\varrho}\frac{z^2}{2!} + \dots,$$

with algebraic coefficients $c_{0\varrho}, c_{1\varrho}, c_{2\varrho}, \dots$. Hence, taking account of (4) and (6),

$$\begin{aligned} c_{01}\tau_1 + c_{02}\tau_2 + \dots + c_{0r}\tau_r &= 0, \\ c_{11}\tau_1 + c_{12}\tau_2 + \dots + c_{1r}\tau_r &= 0, \\ \cdots\cdots\cdots\cdots \end{aligned}$$

Here $\tau_1, \tau_2, \dots, \tau_r$ are linearly independent; it follows therefore

$$\begin{aligned} c_{01} = c_{02} = \dots = c_{0r} &= 0, \\ c_{11} = c_{12} = \dots = c_{1r} &= 0, \\ \cdots\cdots\cdots \end{aligned}$$

or

$$L_\varrho [y(z)] \equiv 0 \ \textit{for} \ \varrho = 1, 2, \dots, r.$$

Every linear differential operator $L_\varrho = b_{0\varrho} + b_{1\varrho} D + \dots + b_{k\varrho} D^k$ has algebraic coefficients; there is at least one whose coefficient $b_{k\varrho}$ does not vanish (on account of (5) and $b_k \neq 0$). Let $\bar{b}_0 + \bar{b}_1 D + \dots + \bar{b}_k D^k$ denote this operator. *Hence $y(z)$ satisfies a linear differential equation*

$$\bar{b}_0 y(z) + \bar{b}_1 y'(z) + \dots + \bar{b}_k y^{(k)}(z) = 0 \tag{7}$$

with algebraic coefficients $\bar{b}_0, \bar{b}_1, \dots, \bar{b}_k$ and $\bar{b}_k \neq 0$.

3. The auxiliary equation

$$\bar{b}_0 + \bar{b}_1 t + \dots + \bar{b}_k t^k = 0$$

of (7) clearly has algebraic roots, say $\bar{\varrho}_1, \bar{\varrho}_2, \dots, \bar{\varrho}_l$; let $\mu_1, \mu_2, \dots, \mu_l$ denote their respective multiplicities. Hence

$$\mu_1 + \mu_2 + \dots + \mu_l = k,$$

and we can write

$$y(z) = \sum_{\lambda=1}^{l} \bar{P}_\lambda(z)\, e^{\bar{\varrho}_\lambda z}, \tag{8}$$

where $\bar{P}_1(z), \bar{P}_2(z), \ldots, \bar{P}_l(z)$ are polynomials. We shall prove, that these polynomials have algebraic coefficients.

The general solution

$$\eta(z) = \gamma_0 + \gamma_1 \frac{z}{1!} + \gamma_2 \frac{z^2}{2!} + \cdots \tag{9}$$

of the equation (7) is given by

$$\eta(z) = \sum_{\lambda=1}^{l} \Pi_\lambda(z)\, e^{\bar{\varrho}_\lambda z} = \sum_{\lambda=1}^{l} \Pi_\lambda(z) \left(1 + \bar{\varrho}_\lambda \frac{z}{1!} + \bar{\varrho}_\lambda^2 \frac{z^2}{2!} + \ldots\right), \tag{10}$$

where

$$\Pi_\lambda(z) = p_{\lambda 0} + p_{\lambda 1} z + \ldots + p_{\lambda, \mu_\lambda - 1} z^{\mu_\lambda - 1}$$

are polynomials.

Equating corresponding coefficients in (9) and (10) we obtain

$$\left\{ \begin{aligned} \gamma_0 &= \sum_{\lambda=1}^{l} p_{\lambda 0} \\ \frac{\gamma_1}{1!} &= \sum_{\lambda=1}^{l} \left(p_{\lambda 0} \frac{\bar{\varrho}_\lambda}{1!} + p_{\lambda 1}\right) \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \frac{\gamma_{k-1}}{(k-1)!} &= \sum_{\lambda=1}^{l} \left(p_{\lambda 0} \frac{\bar{\varrho}_\lambda^{k-1}}{(k-1)!} + p_{\lambda 1} \frac{\bar{\varrho}_\lambda^{k-2}}{(k-2)!} + \ldots\right). \end{aligned} \right. \tag{11}$$

This constitutes for given $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$ a system of k linear equations in the $\mu_1 + \mu_2 + \ldots + \mu_l = k$ quantities $p_{\lambda 0}, p_{\lambda 1}, \ldots, p_{\lambda, \mu_\lambda - 1}$ $(\lambda = 1, 2, \ldots, l)$.

For $\gamma_0 = \gamma_1 = \ldots = \gamma_{k-1} = 0$ the differential equation (7) has only the solution $\eta(z) \equiv 0$; hence in this case the (homogeneous) system (11) has only the trivial solution

$$p_{\lambda 0} = p_{\lambda 1} = \ldots = p_{\lambda, \mu_\lambda - 1} = 0 \qquad (\lambda = 1, 2, \ldots, l).$$

It follows, that the determinant of the system (11)

$$\begin{vmatrix} 1 & 0 & \ldots & 1 & 0 & \ldots & 1 & 0 & \ldots \\ \frac{\bar{\varrho}_1}{1!} & 1 & \ldots & \frac{\bar{\varrho}_2}{1!} & 1 & \ldots & \frac{\bar{\varrho}_l}{1!} & 1 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\bar{\varrho}_1^{k-1}}{(k-1)!} & \frac{\bar{\varrho}_1^{k-2}}{(k-2)!} & \ldots & \frac{\bar{\varrho}_2^{k-1}}{(k-1)!} & \frac{\bar{\varrho}_2^{k-2}}{(k-2)!} & \ldots & \frac{\bar{\varrho}_l^{k-1}}{(k-1)!} & \frac{\bar{\varrho}_l^{k-2}}{(k-2)!} & \ldots \end{vmatrix}$$

does not vanish. Moreover all elements of this determinant are algebraic numbers. In the special case, that $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$ are equally algebraic, the solution $p_{\lambda 0}, p_{\lambda 1}, \ldots, p_{\lambda, \mu_\lambda - 1}$ of (11) necessarily must consist of algebraic numbers. Taking for $\gamma_0, \gamma_1, \ldots, \gamma_{k-1}$ the algebraic coefficients $c_0, c_1, \ldots, c_{k-1}$ of $y(z)$ we obtain as the solution exactly the coefficients of the polynomials $\overline{P}_1(z), \overline{P}_2(z), \ldots, \overline{P}_l(z)$ from (8). Hence: *In*

$$y(z) = \sum_{\lambda=1}^{l} \overline{P}_\lambda(z)\, e^{\overline{\varrho}_\lambda z} \tag{8}$$

$\overline{\varrho}_1, \overline{\varrho}_2, \ldots, \overline{\varrho}_l$ *are different algebraic numbers and the polynomials*

$$\overline{P}_1(z), \overline{P}_2(z), \ldots, \overline{P}_l(z)$$

have algebraic coefficients.

4. We observe that the righthand-sides of (3) and (8) are identical functions of z. Using the elementary theorem b) we may suppose, without loss of generality,

$$\varrho_i = \overline{\varrho}_i, \qquad P_i(z) \equiv \overline{P}_i(z) \not\equiv 0 \qquad (i = 1, 2, \ldots, j) \tag{12}$$

and

$$P_\sigma(z) \equiv 0 \ (\sigma = j+1, j+2, \ldots, s), \quad \overline{P}_\lambda(z) \equiv 0 \ (\lambda = j+1, j+2, \ldots, l)$$

for a positive integer $j \leqq \mathrm{Min}\,(s, l)$. Hence

$$y(z) = \sum_{i=1}^{j} P_i(z)\, e^{\varrho_i z}.$$

By definition $\varrho_1, \varrho_2, \ldots, \varrho_j$ represent zeros of multiplicities $\nu_1, \nu_2, \ldots, \nu_j$ of the characteristic function $a_0 + a_1 t + a_2 t^2 + \ldots$ in the circle $|t| \leqq q$. Moreover every $P_i(z)$ is a polynomial of degree $\nu_i - 1$ at most (see section 1 of this proof). On the other hand by (12) and section 3 the numbers $\varrho_1, \varrho_2, \ldots, \varrho_j$ and the coefficients of $P_1(z), P_2(z), \ldots, P_j(z)$ are all algebraic.

This completes the proof of our lemma.

Proof of theorem I. Without loss of generality we may assume $y(z) \not\equiv 0$. Obviously the function $y(z)$ considered here fulfills all conditions of the preceding lemma. Hence

$$y(z) \equiv \sum_{i=1}^{j} P_i(z)\, e^{\varrho_i z}, \tag{13}$$

where $\varrho_1, \varrho_2, \ldots, \varrho_j$ denote different algebraic zeros of the characteristic function in the circle $|t| \leqq q$; moreover every $P_i(z)$ represents a polynomial with algebraic coefficients of degree $\nu_i - 1$ at most, ν_i denoting the multiplicity of the zero ϱ_i.

Let $\zeta \neq 0$ be an exceptional point of multiplicity μ; hence

$$\zeta, y(\zeta), y'(\zeta), \ldots, y^{(\mu-1)}(\zeta)$$

are algebraic simultaneously. From (13) it follows

$$
(14)\quad \begin{cases}
-y(\zeta)\, e^0 + \sum_{i=1}^{j} P_i(\zeta)\, e^{\varrho_i \zeta} & = 0 \\
-y'(\zeta)\, e^0 + \sum_{i=1}^{j} (P_i'(\zeta) + \varrho_i P_i(\zeta))\, e^{\varrho_i \zeta} & = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots & \\
-y^{(\mu-1)}(\zeta)\, e^0 + \sum_{i=1}^{j} \Big(P_i^{(\mu-1)}(\zeta) + \binom{\mu-1}{1} \varrho_i P_i^{(\mu-2)}(\zeta) + \ldots + & \\
\qquad + \binom{\mu-1}{\mu-1} \varrho_i^{\mu-1} P_i(\zeta) \Big)\, e^{\varrho_i \zeta} = 0. &
\end{cases}
$$

Now we are in a position to apply the LINDEMANN–WEIERSTRASS theorem on each of the μ equations of (14). For in these equations the "exponents" $\varrho_1\zeta, \varrho_2\zeta, \ldots, \varrho_j\zeta$ are different algebraic numbers and the "coefficients"

$$
-y(\zeta), P_i(\zeta) \;\; ; \;\; -y'(\zeta), (P_i'(\zeta) + \varrho_i P_i(\zeta)) \;\; ; \;\; \ldots ;
$$
$$
-y^{(\mu-1)}(\zeta), \Big(P_i^{(\mu-1)}(\zeta) + \binom{\mu-1}{1} \varrho_i P_i^{(\mu-2)}(\zeta) + \ldots + \binom{\mu-1}{\mu-1} \varrho_i^{\mu-1} P_i(\zeta) \Big)
$$

all are algebraic.

For the application of the said theorem it is convenient to consider two special cases separately:

Case I. *Let all numbers $\varrho_1, \varrho_2, \ldots, \varrho_j$ be different from zero.* Then it follows from (14)

$$
\begin{aligned}
& y(\zeta) = 0, \; P_i(\zeta) = 0 \\
& y'(\zeta) = 0, \; P_i'(\zeta) + \varrho_i P_i(\zeta) = 0 \\
& \cdots\cdots\cdots\cdots\cdots\cdots \\
& y^{(\mu-1)}(\zeta) = 0, \; P_i^{(\mu-1)}(\zeta) + \binom{\mu-1}{1} \varrho_i P_i^{(\mu-2)}(\zeta) + \ldots + \binom{\mu-1}{\mu-1} \varrho_i^{\mu-1} P_i(\zeta) = 0
\end{aligned}
$$

for $i = 1, 2, \ldots, j$, hence

$$
(15a) \qquad y(\zeta) = y'(\zeta) = \ldots = y^{(\mu-1)}(\zeta) = 0
$$

and

$$
(15b) \qquad P_i(\zeta) = P_i'(\zeta) = \ldots = P_i^{(\mu-1)}(\zeta) = 0 \qquad (i = 1, 2, \ldots, j).
$$

Case II. *Let one of the numbers $\varrho_1, \varrho_2, \ldots, \varrho_j$ vanish; say* $\varrho_1 = 0$. Then we obtain from (14) in the same manner

$$
\begin{aligned}
& -y(\zeta) + P_1(\zeta) = 0, \; P_i(\zeta) = 0 \\
& -y'(\zeta) + P_1'(\zeta) = 0, \; P_i'(\zeta) + \varrho_i P_i(\zeta) = 0 \\
& \cdots\cdots\cdots\cdots\cdots\cdots \\
& -y^{(\mu-1)}(\zeta) + P_1^{(\mu-1)}(\zeta) = 0, \; P_i^{(\mu-1)}(\zeta) + \binom{\mu-1}{1} \varrho_i P_i^{(\mu-2)}(\zeta) + \ldots + \\
& \qquad\qquad + \binom{\mu-1}{\mu-1} \varrho_i^{\mu-1} P_i(\zeta) = 0
\end{aligned}
$$

for $i = 2, 3, \ldots, j$, hence

$$P_i(\zeta) = P_i'(\zeta) = \ldots = P_i^{(\mu-1)}(\zeta) = 0 \qquad (i = 2, 3, \ldots, j). \tag{16}$$

By hypothesis ν is the largest multiplicity of the zeros of the characteristic function in the region $0 < |t| \leqq q$; hence in case I every polynomial $P_i(z)$ is at most of degree $\nu_i - 1 \leqq \nu - 1$ for $i = 1, 2, \ldots, j$ and in case II the same assertion holds for $i = 2, 3, \ldots, j$.

Now we prove the two assertions of our theorem:

1. Suppose there exist exceptional points $\zeta_1, \zeta_2, \ldots, \zeta_k$ different from zero and with multiplicities $\mu_1, \mu_2, \ldots, \mu_k$, such that

$$\mu_1 + \mu_2 + \ldots + \mu_k \geqq \nu.$$

We have to show, that $y(z)$ is a polynomial with algebraic coefficients.

In case I every polynomial $P_i(z)$ is at most of degree $\nu - 1$ $(i = 1, 2, \ldots, j)$; on the other hand applying (15b) with $\zeta = \zeta_\varkappa$ and $\mu = \mu_\varkappa$ $(\varkappa = 1, 2, \ldots, k)$ we see, that every polynomial $P_i(z)$ at least has ν zeros, hence $P_i(z) \equiv 0$ and therefore $y(z) \equiv 0$, but this gives a contradiction, for we assumed $y(z) \not\equiv 0$.

In case II we similarly apply (16) in stead of (15b) and we obtain $P_i(z) \equiv 0$ for $i = 2, 3, \ldots, j$, hence $y(z) \equiv P_1(z)$, a polynomial with algebraic coefficients.

2. If $a_0 \neq 0$, then $t = 0$ is not a zero of the characteristic function $a_0 + a_1 t + a_2 t^2 + \ldots$, hence $\varrho_1, \varrho_2, \ldots, \varrho_j$ all are different from zero and we have case I. If $\zeta \neq 0$ is an exceptional point with multiplicity μ, then we derive from (15a), that ζ necessarily is a zero of $y(z)$ with multiplicity μ (for $y^{(\mu)}(\zeta)$ is transcendental and therefore different from zero).

§ 2. In 1939 McMillan stated the following theorem, closely related to our theorem:

"Given the set of algebraic numbers a_n $(n = 0, 1, 2, \ldots)$ of which an infinite number are non-vanishing, and such that

$$\limsup_{n\to\infty} |a_n|^{1/n} \leqq \varrho.$$

Let there be a set of constants c_n $(n = 0, 1, 2, \ldots)$, at least two of which are non-vanishing, such that the function

$$g(t) \equiv \sum_{n=0}^{\infty} c_n t^n$$

is analytic for $|t| < R$ where $R > \varrho$, and has zeros for $|t| \leqq \varrho$ only at the points $t = \zeta_k$ $(k = 1, 2, \ldots, N)$ respectively of multiplicities ν_k, where the ζ_k are algebraic numbers. If now

$$\sum_{n=0}^{\infty} c_n a_{n+p} = 0$$

for all $p = 0, 1, 2, \ldots$, then the function

$$F(z) \equiv \sum_{n=0}^{\infty} \frac{a_n z^n}{n!}$$

takes on a transcendental value for every algebraic $z \neq 0$."

However this assertion certainly is not true, as is easily seen from the following example: Take

$a_n = n - 1$ $(n = 0, 1, 2, \ldots)$, $\varrho = 1$, $c_0 = 1$, $c_1 = -2$, $c_2 = 1$, $c_3 = c_4 = \ldots = 0$.

Then it follows

$g(t) \equiv 1 - 2t + t^2$, $N = 1$, $\zeta_1 = 1$, $c_0 a_p + c_1 a_{p+1} + c_2 a_{p+2} = 0 \quad (p = 0, 1, 2, \ldots)$,

so that all hypotheses of McMillan's theorem are satisfied. But now

$$F(z) \equiv \sum_{n=0}^{\infty} (n-1) \frac{z^n}{n!},$$

hence

$$F(1) = 0,$$

not a transcendental number.

§ 3. *A*) Our theorem I clearly is a generalization of the following result of Itihara (see the joint paper of Itihara and Ôishi in the list of references): Let the transcendental function

$$y(z) = \sum_{h=0}^{\infty} c_h \frac{z^h}{h!}$$

with algebraic coefficients $c_0, c_1, c_2, \ldots$ satisfy a linear differential equation

$$a_0\, y(z) + a_1\, y'(z) + \ldots + a_n\, y^{(n)}(z) = 0,$$

with constant algebraic coefficients $a_0, a_1, \ldots, a_n$, then $y(z)$ is a transcendental number for every algebraic value of z with exception of n values for z at most.

B) We can apply theorem I to the differential equation

$$y(z) - y^{(n)}(z) = 0.$$

Obviously the n functions

$$y_\nu(z) = \frac{z^\nu}{\nu!} + \frac{z^{\nu+n}}{(\nu+n)!} + \frac{z^{\nu+2n}}{(\nu+2n)!} + \cdots \qquad (\nu = 0, 1, \ldots, n-1)$$

are all integrals of this equation. Hence every function

$$y(z) = c_0 + c_1 \frac{z}{1!} + c_2 \frac{z^2}{2!} + \cdots,$$

where $c_0, c_1, c_2, \ldots$ constitute a periodic sequence of complex numbers with a "length" n of the period, satisfies the equation. If moreover all coefficients $c_0, c_1, c_2, \ldots$ are algebraic and do not vanish simultaneously, then we can apply the Remark to theorem I (the zeros of the corresponding characteristic function $1 - t^n$ being simple). It follows, that $y(z)$ is a transcendental number for every algebraic value of $z \neq 0$.

From this we easily obtain the following theorem of DIETRICH and ROSENTHAL:

If the coefficients c_h *in*

$$y(z) = \sum_{h=0}^{\infty} c_h \frac{z^h}{h!}$$

are algebraic and form a periodic sequence from some h on, then $y(z)$ *is a transcendental number for every algebraic value of* $z \neq 0$, *except in the trivial case that* $y(z)$ *is a polynomial.*

C) We now show the following theorem of R. RADO:

Suppose that the real functions $f_1(x), f_2(x), \ldots, f_n(x)$, *not all identically zero, of the real variable x satisfy the system of differential equations*

$$f_r'(x) = \sum_{s=1}^{n} c_{rs} f_s(x) \qquad (r = 1, 2, \ldots, n) \tag{17}$$

in which the coefficients c_{rs} *are rational numbers satisfying*

$$\begin{vmatrix} c_{11} & c_{12} & \cdot & \cdot & \cdot & c_{1n} \\ c_{21} & \cdot & \cdot & \cdot & \cdot & c_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n1} & \cdot & \cdot & \cdot & \cdot & c_{nn} \end{vmatrix} \neq 0.$$

Then, for every rational number x_0 *with possibly a single exception, at least one of the numbers*

$$f_1(x_0), f_2(x_0), \ldots, f_n(x_0)$$

is irrational.

Proof. Let α and β be different rational numbers, and assume, that the 2n numbers

$$f_1(\alpha), f_2(\alpha), \ldots, f_n(\alpha); \; f_1(\beta), f_2(\beta), \ldots, f_n(\beta)$$

are rational. Then we will obtain a contradiction.

Without loss of generality we may suppose $\alpha = 0$ and $f_1(x) \not\equiv 0$.

From (17) it follows for $h = 0, 1, 2, \ldots$

$$f_r^{(h+1)}(x) = \sum_{s=1}^{n} c_{rs} f_s^{(h)}(x),$$

where the coefficients c_{rs} are rationals; hence

$$\left.\begin{matrix} f_r(0) & , & f_r'(0) & , & f_r''(0) & , & \ldots \\ f_r(\beta) & , & f_r'(\beta) & , & f_r''(\beta) & , & \ldots \end{matrix}\right\} \quad (r = 1, 2, \ldots, n)$$

all are rational numbers.

From (17) it follows on the other hand by a well-known method, that the n functions $f_1(x), f_2(x), \dots, f_n(x)$ all are integrals of a linear differential equation with constant coefficients

$$a_0\, y(x) + a_1\, y'(x) + \dots + a_n\, y^{(n)}(x) = 0,$$

where $a_0 = \det. |c_{rs}| \neq 0$ and $a_n = (-1)^n$.

Now we are in a position to apply theorem I on the integral function

$$f_1(x) = f_1(0) + f_1'(0)\,\frac{x}{1!} + f_1''(0)\,\frac{x^2}{2!} + \dots,$$

with rational coefficients and with an exceptional argument β of infinite order. By the second assertion of this theorem we obtain

$$f_1(\beta) = f_1'(\beta) = f_1''(\beta) = \dots = 0,$$

hence $f_1(x) \equiv 0$; a contradiction.

Obviously it now is easy to extend RADO's theorem.

D) It is possible to apply theorem I to the solutions of certain functional equations. As an example I consider in the next theorem a certain class of solutions of a linear differential-difference equation with constant coefficients.

Theorem II [3]). *Let the integral transcendental function*

$$y(z) = c_0 + c_1\,\frac{z}{1!} + c_2\,\frac{z^2}{2!} + \dots\ , \quad c_h = 0(q^h), \tag{18}$$

where q is an arbitrary positive number and where all coefficients $c_0, c_1, c_2, \dots$ are algebraic, satisfy a linear differential-difference equation

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A_{\mu\nu}\, y^{(\mu)}(z + \omega_\nu) = 0, \tag{19}$$

where the constants $A_{\mu\nu}$ do not vanish simultaneously and where $\omega_0, \omega_1, \dots, \omega_n$ are different numbers.

Then $y(z)$ is a transcendental number for every algebraic value of z with exception of a finite number of values for z.

If moreover $\sum_{\nu=0}^{n} A_{0\nu} \neq 0$, then these exceptional points, which differ from 0, necessarily are zeros of $y(z)$.

Proof. We have for $\mu = 0, 1, \dots, m$ and $\nu = 0, 1, \dots, n$

$$y^{(\mu)}(z + \omega_\nu) = y^{(\mu)}(z) + \frac{\omega_\nu}{1!}\, y^{(\mu+1)}(z) + \frac{\omega_\nu^2}{2!}\, y^{(\mu+2)}(z) + \dots.$$

Substitution in (19) gives a linear differential equation of infinite order

$$a_0\, y(z) + a_1\, y'(z) + a_2\, y''(z) + \dots = 0,$$

[3]) This theorem was communicated without proof on September 1, 1950, at the International Congress of Mathematicians, Cambridge (Mass.).

with characteristic function

$$a_0 + a_1 t + a_2 t^2 + \dots = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A_{\mu\nu} \left(t^\mu + \frac{\omega_\nu}{1!} t^{\mu+1} + \frac{\omega_\nu^2}{2!} t^{\mu+2} + \dots \right)$$

$$= \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A_{\mu\nu} t^\mu e^{\omega_\nu t}.$$

Clearly

$$a_0 = \sum_{\nu=0}^{n} A_{0\nu}.$$

Now $y(z)$ fulfills all conditions of theorem I and the assertions of theorem II are immediate consequences of those of theorem I.

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